Math 181: Problem Set #6

Jesse Kass

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Due in 1 week at the start of class. Make sure to read Chapters 1 and 2 of Wardhaugh's *How to Read Historical Mathematics* and Chapter 4, "Learning mathematics," of the Stedall book.

Problem 1

For each of the following numbers, state whether or not it can be described in terms of Archimedes number system of orders and period. If it can be described, describe it.

1. 20988936657440586486151264256610222593863921 (the largest prime number that was found before the use of electronic computer)
2. 200,000,000 (the amount of time it took a fast computer to compute a 10-digit number circa 1970).
3. 340,282,366,939,463,463,374,607,431,768,211,457 (a very large number that was factored in 1970).
4. $2^{82589933} - 1$ (the largest known Mersenne prime).

Problem 2

Beginning with the sentence that starts "Es ist aber auch," Figures 1 and 2 contain a German translation of Archimedes' description of essentially the law of exponents ($x^a \cdot x^b = x^{a+b}$). Translate the text using whatever resources you want (Google Translate, a German friend,....).

Then write a paragraph comparing and contrasting Heath's English translation of Archimedes. The relevant text by Heath appears below as Figures 3 and 4.

Problem 3

The attached article "Coping with finiteness" by Donald Knuth includes a description up-arrow notation for large numbers. Using the up-arrow notation, describe

1. a number that can be described using Archimedes' system of orders and periods and has period at least $\overline{\gamma}$.
2. a number that is so large it cannot be described using Archimedes' system of orders and periods

Problem 4

Write a review of the book *The Archimedes Codex* that is written for a future student in Math 181. Would you advise reading this book? Why or why not? What does the book do well? What does it do poorly. The review should be at least a half-page long.

Problem 5

Read the "Essay Guidelines" (to be distributed) for the final essay for the class. Find at least two primary sources that would you be interested in studying for your essay. What are they? Provide enough details that the grader can find the sources. Why did you pick those sources?

Collaboration Policy

With each week's homework, you must turn in a one paragraph description of all the resources you used on that homework. You must mention any person you talked to about the problems, any book you looked at, any online resource (Wikipedia, Chegg,...) that you used. A sample paragraph is

> On this week's homework, I worked on the problem set collaboratively with Gauss and Grothendieck at The Redroom during happy hour. We found an Alex Jones video (http://youtube.blah.com) that gave a really clear explanation of Fermat's Last Theorem. We got really stuck on Problem 5, and so we went to Chegg.com and paid an online tutor ("Zariski") $50 to solve the problem for us. He said the problem was too hard for him. So I logged into my TruthSocial account (@CobraTatesThesis) and posted the question with @realDonaldTrump tagged. He responded with a tremendous, really fantastic solution to the problem, which by the way, Biden can't solve. At this point, it was midnight and I still had four more problems to go, so I just gave the questions to ChatGPT and cut-and-pasted the answers.

100 000 000 e_2 und so fort. Es ist aber auch nützlich, folgendes zu erkennen: Wenn eine geometrische Reihe vorhanden ist, deren Glieder mit a_1, a_2, a_3 . . . . bezeichnet werden und deren Anfangsglied $a_1 = 1$ ist, so ist $a_m \cdot a_n$ 7
$= a_{m+n-1}$. Es sei z. B. $A\,B\,C\,D\,E\,F\,G\,H\,I\,K\,L$ eine solche Reihe. A sei gleich 1, und es möge D mit H multipliziert werden. Das Produkt sei X. Es möge dann die Zahl bestimmt werden, die von H so weit entfernt ist wie D von A. Dies ist die Zahl L. Dann ist zu zeigen, daß $X = L$ ist. Da nämlich $D : H = L : H$ ist, so ist $DH = LA$, und da $LA = A$ ist, so ist $L = DH$, also $X = L$. Es ist klar, daß das Produkt innerhalb der Reihe vom größeren Faktor so weit entfernt ist wie der kleinere von der Ein-

Figure 1: Part of a German translation of Archimedes' *The Sand Reckoner*

heit. Es ist aber auch klar, daß das Produkt von der Einheit um 1 Glied weniger weit entfernt ist, als die Summe der Entfernungen der beiden Faktoren von der Einheit beträgt. Denn L ist von A aus das 10. Glied, D von A aus das 4. und H von A aus das 7. Glied.

Figure 2: The German translation continued

Theorem.

If there be any number of terms of a series in continued proportion, say $A_1, A_2, A_3, \dots A_m, \dots A_n, \dots A_{m+n-1}, \dots$ of which **$A_1 = 1$, $A_2 = 10$ [so that the series forms the geometrical progression $1, 10^1, 10^2, \dots 10^{m-1}, \dots 10^{n-1}, \dots 10^{m+n-2}, \dots$],** ***and if any two terms as A_m, A_n be taken and multiplied, the product***

Figure 3: Part of an English translation of Archimedes' *The Sand Reckoner*

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$A_m . A_n$ will be a term in the same series and will be as many terms distant from A_n as A_m is distant from A_1; also it will be distant from A_1 by a number of terms less by one than the sum of the numbers of terms by which A_m and A_n respectively are distant from A_1.

Take the term which is distant from A_n by the same number of terms as A_m is distant from A_1. This number of terms is m (the first and last being both counted). Thus the term to be taken is m terms distant from A_n, and is therefore the term A_{m+n-1}.

We have therefore to prove that

$$A_m . A_n = A_{m+n-1}.$$

Now terms equally distant from other terms in the continued proportion are proportional.

Thus $$\frac{A_m}{A_1} = \frac{A_{m+n-1}}{A_n}.$$

But $$A_m = A_m . A_1, \text{ since } A_1 = 1.$$

Therefore $$A_{m+n-1} = A_m . A_n \qquad \text{(1)}.$$

The second result is now obvious, since A_m is m terms distant from A_1, A_n is n terms distant from A_1, and A_{m+n-1} is $(m+n-1)$ terms distant from A_1.

Figure 4: The English translation continued

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SCIENCE

Mathematics and Computer Science: Coping with Finiteness

Advances in our ability to compute are bringing us substantially closer to ultimate limitations.

Donald E. Knuth

A well-known book entitled *One, Two, Three, . . . Infinity* was published by Gamov about 30 years ago (*1*), and he began by telling a story about two Hungarian noblemen. It seems that the two gentlemen were out riding, and one suggested to the other that they play a game: Who can name the largest number. "Good," said the second man, "you go first." After several minutes of intense concentration, the first nobleman announced the largest number he could think of: "Three." Now it was the other man's turn, and he thought furiously, but after about a quarter of an hour he gave up. "You win," he said.

In this article I will try to assess how much further we have come, by discussing how well we can now deal with large quantities. Although we have certainly narrowed the gap between three and infinity, recent results indicate that we will never actually be able to go very far in practice. My purpose is to explore relationships between the finite and the infinite, in the light of these developments.

Some Large Finite Numbers

Since the time of Greek philosophy, men have prided themselves on their ability to understand something about infinity; and it has become traditional in some circles to regard finite things as essentially trivial, too limited to be of any interest. It is hard to debunk such a notion, since there are no accepted standards for demonstrating that something is interesting, especially when something finite is compared with something transcendent. Yet I believe that the climate of thought is changing, since finite processes are proving to be such fascinating objects of study.

In the first place, it is important to understand that finite numbers can be extremely large. Let us start with some very familiar and fairly small numbers: the value of xn is $x + x + \cdots + x$, added n times. Similarly we can define a number I shall write as $x \uparrow n$, which means $xx \cdots x$ multiplied n times. For example, $10 \uparrow 10 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000{,}000{,}000$ is 10 billion; this is usually written 10^{10}, but it will be clear in a minute why I prefer to use an upward arrow. In fact, the next step uses two arrows

$$x \uparrow\uparrow n = x \uparrow (x \uparrow (\cdots \uparrow x) \cdots))$$

where we take powers n times. For example

$$10 \uparrow\uparrow 10 = 10^{10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}}$$

$$= 1 \text{ followed by } 10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}} \text{ zeros}$$

This is a pretty big number; at least, if a monkey sits at a typewriter and types at random, the average number of trials before he types perfectly the entire text of Shakespeare's *Hamlet* would be much, much less than this: it is merely a 1 followed by about 40,000 zeros. The general rule is

$$x \overbrace{\uparrow\uparrow \ldots \uparrow}^{k \text{ arrows}} n = \underbrace{x \overbrace{\uparrow \ldots \uparrow}^{k-1} (x \overbrace{\uparrow \ldots \uparrow}^{k-1} (\cdots \overbrace{\uparrow \ldots \uparrow}^{k-1} x) \cdots))}_{n \text{ times}}$$

Thus, one arrow is defined in terms of none, two in terms of one, three in terms of two, and so on.

In order to see how these arrow functions behave, let us look at a very small example

$$10 \uparrow\uparrow\uparrow\uparrow 3$$

This is equal to

$$10 \uparrow\uparrow\uparrow (10 \uparrow\uparrow\uparrow 10)$$

so we should first evaluate $10 \uparrow\uparrow\uparrow 10$. This is

$$10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow 10))))))))$$

and that is

$$10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow 10^{10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}}))))))))$$

$$= 10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow (10 \uparrow\uparrow 10^{10^{\cdot^{\cdot^{10}}}}))))))$$

where the stack of 10's is $10 \uparrow\uparrow 10$ levels tall. We take the huge number at the right of this formula, which I cannot even write down without using the arrow notation, and repeat the double-arrow operation, getting an even huger number, and then we must do the same thing again and again. Let us call the final

The author is professor of computer science at Stanford University, Stanford, California 94305.

result $\mathscr{H}$. (It is such an immense number, we cannot use just an ordinary letter for it.)

Of course we are not done yet, we have only evaluated $10\uparrow\uparrow\uparrow 10$; to complete the job we need to stick this gigantic number into the formula for $10\uparrow\uparrow\uparrow\uparrow 3$, namely

$$10\uparrow\uparrow\uparrow\uparrow 3 = 10\uparrow\uparrow\uparrow\mathscr{H}$$
$$= \underbrace{10\uparrow\uparrow(10\uparrow\uparrow(10\uparrow\uparrow\cdots\uparrow\uparrow(10\uparrow\uparrow 10)\cdots))}_{\mathscr{H}\text{ times}}$$

The three dots " ··· " here suppress a lot of detail—maybe I should have used four dots. At any rate it seems to me that the magnitude of this number $10\uparrow\uparrow\uparrow\uparrow 3$ is so large as to be beyond human comprehension.

On the other hand, it is very small as finite numbers go. We might have used $\mathscr{H}$ arrows instead of just four, but even that would not get us much further—almost all finite numbers are larger than this. I think this example helps open our eyes to the fact that some numbers are very large even if they are merely finite. Thus, mathematicians who stick mostly to working with finite numbers are not really limiting themselves too severely.

Realistic Numbers

This discussion has set the stage for the next point I want to make, namely that our total resources are not actually very large. Let us try to see how big the known universe is. Archimedes began such an investigation many years ago, in his famous discussion of the number of grains of sand that would completely fill the earth and sky; he did not have the benefit of modern astronomy, but his estimate was qualitatively the same as what we would say today. The distance to the farthest observable galaxies is thought to be at most about 10 billion light years. On the other hand, the fundamental nucleons that make up matter are about 10^{-12} centimeter in diameter. In order to get a generous upper bound on the size of the universe, let us imagine a cube that is 40 billion light years on each side, and fill it with tiny cubes that are smaller than protons and neutrons, say 10^{-13} cm on each side (see Fig. 1). The total number of little cubes comes to less than 10^{125}. We might say that this is an "astronomically large" number, but actually it has only 125 digits.

Instead of talking only about large numbers of objects, let us also consider the time dimension. Here the numbers are much smaller; for example, if we take as a unit the amount of time that

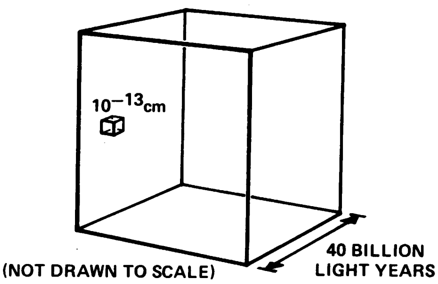

Fig. 1. The known universe fits inside this box.

light rays take to travel 10^{-13} cm, the total number of time units since the dawn of the universe is only one fourth the number of little cubes along a single edge of the big cube in Fig. 1, assuming that the universe is 10 billion years old.

Coming down to earth, it is instructive to consider typical transportation speeds.

Snail	0.006	mile/hour
Man walking	4	mile/hour
U.S. automobile	55	mile/hour
Jet plane	600	mile/hour
Supersonic jet	1200	mile/hour

I would never think of walking from California to Boston, but the plane flight is only 150 times faster. Compare this to the situation with respect to the following computation speeds, given 10-digit numbers.

Man (pencil and paper)	0.2/sec
Man (abacus)	1/sec
Mechanical calculator	4/sec
Medium-speed computer	200,000/sec
Fast computer	200,000,000/sec

A medium-fast computer can add 1 million times faster than we can, and the fastest machines are 1000 times faster yet. Such a ratio of speeds is unprecedented in history: consider how much a mere factor of 10 in speed, provided by

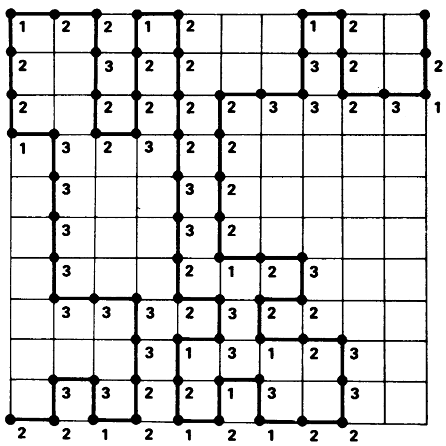

Fig. 2. A "random" path from the lower left corner to the upper right corner of a 10 × 10 grid.

the automobile, has changed our lives, and note that computers have increased our calculation speeds by six orders of magnitude; that is more than the ratio of the fastest airplane velocity to a snail's pace.

I do not mean to claim that computers do everything a million times faster than people can; mere mortals like us can do some things much better. For example, you and I can even recognize the face of a friend who has recently grown a moustache; and for tasks like filing, a computer may be only ten or so times faster than a good secretary. But when it comes to arithmetic, computers appear to be almost infinitely fast compared with people.

As a result, we have begun to think about computational problems that used to be unthinkable. Our appetite for calculation has caused us to deal with finite numbers much larger than those we considered before, and this has opened up a rich vein of challenging problems, just as exciting as the problems about infinity which have inspired mathematicians for so many centuries.

Of course, computers are not infinitely fast, and our expectations have become inflated even faster than our computational capabilities. We are forced to realize that there are limits beyond which we cannot go. The numbers we can deal with are not only finite, they are very finite, and we do not have the time or space to solve certain problems even with the aid of the fastest computers. Thus, the theme of this article is coping with finiteness: What useful things can we say about these finite limitations? How have people learned to deal with the situation?

Advances in Technology and Techniques

During the last 15 years computer designers have made computing machines about 1000 times faster. Mathematicians and computer scientists have also discovered a variety of new techniques, by which many problems can now be solved enormously faster than they could before. I will present several examples of this; the first one, which is somehow symbolic of our advances in arithmetic ability, is the following factorization of a very large number, completed in 1970 by Morrison and Brillhart (2).

340,282,366,920,
938,463,463,374,607,431,768,211,457
= 5,704,689,200,685,129,054,721 ×
59,649,589,127,497,217