Math 201: Problem Set #1

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Due in 1 week.

Problem 1

Write 1 paragraph explaining (1) your background in algebra and (2) your motivation for taking the course.

Problem 2

Let $v = (a_1, a_2)$ and $w = (b_1, b_2)$ be two linearly independent vectors in the plane $\mathbb{R}^2$. The subset of vectors $rv + sw$ with $0 \leq r, s, \leq 1$ is a parallelogram P. The absolute value $A(v, w) = |a_1 b_2 - a_2 b_1|$ equals the area of P. Give *geometric* answers to the following:

1. Permuting the vectors does not change the area: $A(v, w) = A(w, v)$.
2. Scaling one of the vectors scales the area: $A(\lambda v, w) = |\lambda| \cdot A(v, w)$.
3. A "sheaf transformation" leaves the area unchanged: $A(v + w, w) = A(v, w)$.
4. Can every parallelogram be reduced to a unit square using a sequence of these steps?

Problem 3

Now suppose that $u = (a_1, a_2, a_3)$, $v = (b_1, b_2, b_3)$, and $w = (c_1, c_2, c_3)$ be three linearly independent vectors in the plane $\mathbb{R}^3$. These vectors space a parallelopiped $P(u, v, w)$. Let $V(u, v, w)$ be the volume of P. Develop the properties of $V(u, v, w)$ along the lines of the previous problem. What happens if we replace $\mathbb{R}^3$ with $\mathbb{R}^4$?

Problem 4

Now let's replace the absolute value in problem 1 with a p-adic absolute value. Fix $p \in \mathbb{Z}$ a prime number. For a nonzero integer n, define the p-adic absolute value by

$$|n|_p = p^{-\operatorname{ord}_p(n)}.$$

Here $\operatorname{ord}_p(n)$ is the largest integer k such that p^k divides n. Extend this to zero by setting $|0|_p = 0$ and to rational numbers by setting $|n/m|_p = |n|_p/|m|_p$.

Now define the p-adic area of the parallelogram spanned by two vectors $v, w \in \mathbb{Q}^2$ by $A(v, w) = |a_1 b_2 - a_2 b_1|_p$. This no longer has an (obvious) geometric interpretation. Do the properties from problem 1 remain valid?

Problem 5

Let G be the abelian group generated by three elements u, v, w and the relations $-4u + 2v + 6w = -6u + 2v + 6w = 7u + 4v + 15w = 0$. Does there exist a nonconstant function $V : G \times G \times G \to \mathbb{R}$ satisfying the conditions you came up with in problem 2?

Collaboration Policy

With each week's homework, you must turn in a one paragraph description of all the resources you used on that homework. You must mention any person you talked to about the problems, any book you looked at, any online resource (Wikipedia, Chegg,...) that you used. A sample paragraph is

> On this week's homework, I worked on the problem set collaboratively with Gauss and Grothendieck at The Redroom during happy hour. We found an Alex Jones video (http://youtube.blah.com) that gave a really clear explanation of Fermat's Last Theorem. We compared our solutions against a solution key that we found on the /commutativealgebra/ board of 4chan (http://blah.blah.edu). We also got really stuck on Problem 5, and so we went to Chegg.com and paid an online tutor ("Zariski") $50 to solve the problem for us.