

Math 201: Problem Set #6

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Due in 1 week.

Problem 1

In class and in the textbook, we discussed the adjunction relation $\text{Hom}(A \otimes_R B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$. Here we will explore a similar relation.

1. Let A be a \mathbb{Z} -module. Imitate the construction of the rational numbers \mathbb{Q} from \mathbb{Z} to construct using $A/\text{Tor}(A)$ a module $F(A)$ whose elements are “fractions” a/q with $A/\text{Tor}(A)$ and $q \in \mathbb{Q} - 0$.

(Addition should satisfies $a_1/q_1 + a_2/q_2 = (q_2 a_1 + q_1 a_2)/(q_1 q_2)$ etc).

2. Given a \mathbb{Q} -module B , let $G(B)$ denote B considered as a \mathbb{Z} -module (only allow integers as scalars). Construct an isomorphism

$$\text{Hom}(F(A), B) = \text{Hom}(A, G(B)).$$

Problem 2

Solve Exercise 2 in section 12.1 of Dummit and Foote.

Problem 3

Solve Exercise 15 in section 12.1 of Dummit and Foote.

Problem 4

The text displayed in Figure 1 is Chat GPT’s solution to problem 8 in section 12.1. Is it correct? Read the problem as through you were the grader. Describe any errors and then write up a proof in your own words. (If Chat GPT’s proof is correct, you can just rewrite it.)

Collaboration Policy

With each week's homework, you must turn in a one paragraph description of all the resources you used on that homework. You must mention any person you talked to about the problems, any book you looked at, any online resource (Wikipedia, Chegg,...) that you used. A sample paragraph is

On this week's homework, I worked on the problem set collaboratively with Gauss and Grothendieck at The Redroom during happy hour. We found an Alex Jones video (<http://youtube.blah.com>) that gave a really clear explanation of Fermat's Last Theorem. We compared our solutions against a solution key that we found on the /commutativealgebra/ board of 4chan (<http://blah.blah.edu>). We also got really stuck on Problem 5, and so we went to Chegg.com and paid an online tutor ("Zariski") \$50 to solve the problem for us.

**ChatGPT**

To prove this statement, we'll use the definitions and properties of principal ideal domains (PIDs), torsion modules, and annihilators.

Let R be a principal ideal domain, B a torsion R -module, and p a prime in R . We want to show that if $p \cdot b = 0$ for some nonzero b in B , then $\text{Ann}(B)$ is a subset of the ideal generated by p .

Recall that the annihilator of a module B , denoted $\text{Ann}(B)$, is defined as:

$$\text{Ann}(B) = \{r \in R \mid r \cdot b = 0 \text{ for all } b \in B\}$$

Now, given that $p \cdot b = 0$ for some nonzero b in B , we want to show that $\text{Ann}(B)$ is a subset of the ideal generated by p .

Let's prove this:

1. **Show p generates an ideal:** Since p is a prime element in R , the ideal generated by p , denoted (p) , is a prime ideal.
2. **Show $\text{Ann}(B)$ is an ideal:** $\text{Ann}(B)$ is an ideal of R because it is the kernel of the module homomorphism $\phi : R \rightarrow B$ defined by $\phi(r) = r \cdot b$.
3. **Show $\text{Ann}(B) \subseteq (p)$:** Since $p \cdot b = 0$ for some nonzero b in B , every element in $\text{Ann}(B)$ must be a multiple of p (since p generates a prime ideal). Therefore, $\text{Ann}(B)$ is a subset of the ideal generated by p .

Thus, we have shown that if $p \cdot b = 0$ for some nonzero b in B , then $\text{Ann}(B)$ is a subset of the ideal generated by p , as required.



Figure 1: Chat GPT's solution to problem 8 in section 12.1