ELEMENTS

OF

ALGEBRA:

INCLUDING STURMS' THEOREM.

TRANSLATED FROM THE FRENCH OF

M. BOURDON.

ADAPTED TO THE COURSE OF MATHEMATICAL INSTRUCTION IN THE UNITED STATES,

BY CHARLES DAVIES, LL.D.

AUTHOR OF ARITHMETIC, ELEMENTARY ALGEBRA, ELEMENTARY GEOMETRY, PRACTICAL GEOMETRY, ELEMENTS OF SURVEYING, ELEMENTS OF DESCRIPTIVE AND ANALYTICAL GEOMETRY, ELEMENTS OF DIFFERENTIAL AND INTEGRAL CALCULUS, AND A TREATISE ON SHADES, SHADOWS, AND PERSPECTIVE.

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PREFACE

The Treatise on Algebra, by M. Bourdon, is a work of singular excellence and merit. In France, it is one of the leading text books. Shortly after its first publication, it passed through several editions, and has formed the basis of every subsequent work on the subject of Algebra.

The original work is, however, a full and complete treatise on the subject of Algebra, the later editions containing about eight hundred pages octavo. The time which is given to the study of Algebra, in this country, even in those seminaries where the course of mathematics is the fullest, is too short to accomplish so voluminous a work, and hence it has been found necessary either to modify it, or to abandon it altogether.

The following work is abridged from a translation of M. Bourdon, made by Lieut. Ross, now the distinguished professor of mathematics in Kenyon College, Ohio.

The Algebra of M. Bourdon, however, has been regarded only as a standard or model. The order of arrangement, in many parts, has been changed; new rules and new methods have been introduced; and all the modifications which have been suggested by teaching and a careful comparison with other standard works, have been freely made. It would, perhaps, not be just to regard M. Bourdon as responsible for the work in its present form.

It has been the intention to unite in this work, the scientific discussions of the French, with the practical methods of the English school; that theory and practice, science and art, may mutually aid and illustrate each other.

CHARLES DAVIES.

WEST POINT, *June*, 1844.

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3. What are the entire roots of the equation

$$15x^5 - 19x^4 + 6x^3 + 15x^2 - 19x + 6 = 0?$$

4. What are the entire roots of the equation

$$9x^6 + 30x^5 + 22x^4 + 10x^3 + 17x^2 - 20x + 4 = 0.$$

Sturms' Theorem.

333. The object of this theorem is to explain a method of determining the number and places of the real roots of equations involving but one unknown quantity. Let

$$X = 0 \ldots . (1),$$

represent an equation containing the single unknown quantity x; X being a polynomial of the m^{th} degree with respect to x, the co-efficients of which are all real. If this equation should have equal roots, they may be found and divided out as in Art. 302, and the following reasoning be applied to the equation which would result. We will therefore suppose $X = 0$ to have no equal roots.

334. Let us denote the first-derived polynomial of X by X_1, and then apply to X and X_1 a process similar to that for finding their greatest common divisor, differing only in this respect, that instead of using the successive remainders as at first obtained, *we change their signs, and take care also, in preparing for the division, neither to introduce nor reject any factor except a positive one.*

If we denote the several remainders, in order, after their signs have been changed, by $X_2, X_3 \ldots X_r$, which are read X second, X third, &c., and denote the corresponding quotients by Q_1, Q_2 $\ldots Q_{r-1}$, we may then form the equations

$$X = X_1Q_1 - X_2 \ldots . (2),$$

$$\left.\begin{array}{l} X_1 = X_2Q_2 - X_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_{n-1} = X_nQ_n - X_{n+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X_{r-2} = X_{r-1}Q_{r-1} - X_r \end{array}\right\} \ldots (3).$$

Since by hypothesis, $X = 0$ has no equal roots, no common divisor can exist between X and X_1 (Art. 300). The last remainder $-X_r$, will therefore *be different from zero,* and *independent* of x.

335. Now, let us suppose that a number p has been substituted for x in each of the expressions X, X_1, $X_2 \ldots X_{r-1}$; and that the signs of the results, together with the sign of X_r, are arranged in a line one after the other: also that another number q, greater than p, has been substituted for x, and the signs of the results arranged in like manner.

Then will the number of variations in the signs of the first arrangement, diminished by the number of variations in those of the second, denote the exact number of real roots comprised between p *and* q.

336. The demonstration of this truth mainly depends upon the four following properties of the expressions X, $X_1 \ldots X_n$, &c.

I. Let a be a root of the equation $X = 0$. If we substitute $a + u$ for x, and designate by A what X becomes, and denote the derived polynomials by A', A'', A''', &c.; we shall have (Art. 299),

$$A + A'u + \frac{A''}{2}u^2 \ldots\ldots + u^m.$$

But since by hypothesis, a is a root of the equation $X = 0$, we have $A = 0$, and hence the above expression becomes

$$u\left(A' + \frac{A''}{2}u + \frac{A'''}{2 \, . \, 3}u^2 \ldots\ldots + u^{n-1}\right);$$

in which A' is not zero, since the equation $X = 0$ is supposed not to contain equal roots. Now we say, *that* u *can be made so small, that the sign of the quantity within the parenthesis shall be the same as that of its first term.*

We attain this object, by finding for u a value which shall render, numerically,

$$A' > \frac{A''}{2}u + \frac{A'''}{2 \, . \, 3}u^2 + \text{\&c.} \ldots\ldots u^{n-1};$$

that is, $$A' > u\left(\frac{A''}{2} + \frac{A'''}{2 \, . \, 3}u + \text{\&c.} \ldots\ldots u^{n-2}\right);$$

a condition which will always be fulfilled (Art. 315), when

$$u = \text{ or } < \frac{A'}{K+A'}, \; K \text{ being the greatest co-efficient of } u.$$

II. *If any number be substituted for* x *in these expressions, it is impossible that any two consecutive ones can become zero at the same time.*

For, let X_{n-1}, X_n, X_{n+1}, be any three consecutive expressions. Then among equations (3), we shall find

$$X_{n-1} = X_n Q_n - X_{n+1} \quad \ldots \ldots (4),$$

from which it appears that, if X_{n-1} and X_n should both become 0 for a value of x, X_{n+1} would be 0 for the same value; and since the equation which follows (4) must be

$$X_n = X_{n+1} Q_{n+1} - X_{n+2},$$

we shall have $X_{n+2} = 0$ for the same value, and so on until we should find $X_r = 0$, which cannot be; hence, X_{n-1} and X_n cannot both become 0 for the same value of x.

III. By an examination of equation (4), we see that if X_n becomes 0 for a value of x, X_{n-1} and X_{n+1} must have contrary signs; that is, *if any one of the expressions is reduced to* 0 *by the substitution of a value for* x, *the preceding and following ones will have contrary signs for the same value.*

IV. Let us substitute $a + u$ for x in the expressions X and X_1, and designate by U and U_1 what they respectively become under this supposition. Then (Art. 297), we have

$$\left.\begin{aligned} U &= A + A'u + A''\frac{u^2}{2} + \&c. \\ U_1 &= A_1 + A'_1 u + A''_1 \frac{u^2}{2} + \&c. \end{aligned}\right\} \ldots\ldots (5),$$

in which A, A', A'', &c., are the results obtained by the substitution of a for x, in X and its derived polynomials; and A_1 A'_1 &c., are similar results derived from X_1. If now, a be a root of the proposed equation $X = 0$, then $A = 0$, and since A' and A_1 are each derived from X_1, by the substitution of a for x, we have $A' = A_1$, and equations (5) become

$$\left.\begin{aligned} U &= A'u + A''\frac{u^2}{2} + \&c. \\ U_1 &= A' + A'_1 u \quad + \&c. \end{aligned}\right\} \ldots\ldots (6).$$

Now, the arbitrary quantity u may be taken so small that when added to a, it will but insensibly increase it, and when subtracted from a, it will but insensibly diminish it; in which cases, the signs of the values of U and U_1 will depend upon the signs of their first terms; that is, they will be alike when u is positive or when $a + u$ is substituted for x, and unlike when u is negative or when

$a - u$ is substituted for x. Hence, *if a number insensibly less than one of the real roots of $X = 0$ be substituted for* x *in X and X_1, the results will have contrary signs, and if a number insensibly greater than this root be substituted, the results will have the same sign.*

337. Now, let any number as k, algebraically less, that is, nearer equal to $-\infty$, than any of the real roots of the several equations

$$X = 0, \quad X_1 = 0 \ldots X_{r-1} = 0,$$

be substituted for x in them, and the signs of the several results arranged in order; then, let x be increased by insensible degrees, until it becomes equal to h the least of all the roots of the equations. As there is no root of either of the equations between k and h, none of the signs can change while x is less than h (Art. 311), and the number of variations and permanences in the several sets of results, will remain the same as in those obtained by the first substitution.

When x becomes equal to h, *one* or *more* of the expressions X, X_1, &c., will reduce to 0. Suppose X_n becomes 0. Then, as by the second and third properties above explained, neither X_{n-1} nor X_{n+1} can become 0 at the same time, but *must have* contrary signs, it follows that in passing from one to the other (omitting $X_n = 0$), there will be *one* and *only one* variation · and since their signs have not changed, one must be *the same as*, and the other *contrary to*, that of X_n, both before and after it becomes 0; hence, in passing over the three, either just before X_n becomes 0 or just after, there is *one* and *only one* variation. Therefore, the reduction of X_n to 0 neither increases nor diminishes the number of variations; and this will evidently be the case, although several of the expressions X_1, X_2, &c., should become 0 at the same time.

If $x = h$ should reduce X to 0, then h is the least real root of the proposed equation, which root we denote by a; and since by the fourth property, just before x becomes equal to a, the signs of X and X_1 are contrary, *giving a variation*, and just after passing it (before x becomes equal to a root of $X_1 = 0$), the signs are the same, *giving a permanence instead*, it follows that in passing this root *a variation is lost.* In the same way, increasing x by insensible degrees from $x = a + u$ until we reach the root of $X = 0$ next in order, it is plain that no variation will be lost or gained in passing any of the roots of the other equations, but that in passing this root, for the same reason as before, another variation will be lost, and so on for each real root between k and the number last substituted, as g, a variation will be lost until x has been increased beyond the greatest real root, *when no more can be lost or gained.* Hence, the *excess* of the number of variations obtained by the substitution of k over those obtained by the substitution of g, will be equal to the number of real roots comprised between k and g.

It is evident that the same course of reasoning will apply when we commence with any number p, whether less than all the roots or not, and gradually increase x until it equals any other number q. The fact enunciated in Art. 335 is therefore established.

338. In seeking the number of roots comprised *between* p and q, should either p or q reduce any of the expressions X_1, X_2, &c., to 0, the result will not be affected by their omission, since the number of variations will be the same.

Should p reduce X to 0, then p is a root, but not one of those sought; and as the substitution of $p + u$ will give X and X_1 the same sign, the number of variations to be counted will not be affected by the omission of $X = 0$.

Should q reduce X to 0, then q is also a root; and as the substitution of $q - u$ will give X and X_1 contrary signs, one variation must be counted in passing from X to X_1.

339. If in the application of the preceding principles, we observe that any one of the expressions X_1, X_2 . . . &c., X_n for instance, will preserve the same sign for all values of x in passing from p to q, inclusive, it will be unnecessary to use the succeeding expressions, or even to deduce them. For, as X_n preserves the same sign during the successive substitutions, it is plain that the same number of variations will be lost among the expressions X, X_1, &c. . . . ending with X_n as among all including X_r. Whenever then, in the course of the division, it is found that by placing any of the remainders equal to 0, an equation is obtained with imaginary roots only (Art. 325), it will be useless to obtain any of the succeeding remainders. This principle will be found very useful in the solution of numerical examples.

340. As all the real roots of the proposed equation are necessarily included between $-\infty$ and $+\infty$, we may, by ascertaining

the number of variations lost by the substitution of these, in succession, in the expressions X, $X_1 \ldots X_n$, . . &c., readily determine the total number of such roots. It should be observed, that it will be only necessary to make these substitutions in the first terms of each of the expressions, as in this case the sign of the term will determine that of the entire expression (Art. 315).

341. Having thus obtained the total number of real roots, we may ascertain their places by substituting for x, in succession, the values 0, 1, 2, 3, &c., until we find an entire number which gives the same number of variations as $+\infty$. This will be the smallest superior limit of the positive roots in entire numbers.

Then substitute 0, -1, -2, &c., until a negative number is obtained which gives the same number of variations as $-\infty$ This will be, numerically, the smallest superior limit of the negative roots in entire numbers. Now, by commencing with this limit and observing the number of variations lost in passing from each number to the next in order, we shall discover how many roots are included between each two of the consecutive numbers used, and thus, of course, know the entire part of each root. The decimal part may then be sought by some of the known methods of approximation.

EXAMPLES.

1. Let $$8x^3 - 6x - 1 = 0 = X.$$

The first-derived polynomial (Art. 297), is

$$24x^2 - 6,$$

and since we may omit the positive factor 6, without affecting the sign, we may write

$$4x^2 - 1 = X_1.$$

Dividing X by X_1, we obtain for the first remainder, $-4x - 1$. Changing its sign, we have

$$4x + 1 = X_2.$$

Multiplying X_1 by the positive number 4, and then dividing by X_2, we obtain the second remainder -3; and by changing its sign

$$+3 = X_3.$$

The expressions to be used are then

$$X = 8x^3 - 6x - 1, \quad X_1 = 4x^2 - 1, \quad X_2 = 4x + 1, \quad X_3 = +3.$$

Substituting $-\infty$ and then $+\infty$, we obtain the two following arrangements of signs:

$- + - +$ . . . . . . 3 *variations*,

$+ + + +$ . . . . . . 0 "

There are then *three* real roots.

If now, in the same expressions we substitute 0 and $+1$, and then 0 and -1, for x, we shall obtain the three following arrangements:

For	$x = +1$	$+ + + +$	0 *variations*,
"	$x = 0$	$- - + +$	1 "
"	$x = -1$	$- + - +$	3 "

As $x = +1$ gives the same number of variations as $+\infty$, and $x = -1$ gives the same as $-\infty$, $+1$ and -1 are the smallest limits in entire numbers. In passing from -1 to 0, *two* variations are lost, and in passing from 0 to $+1$, *one* variation is lost; hence, there are two negative roots between -1 and 0, and one positive root between 0 and $+1$.

2. Let $2x^4 - 13x^2 + 10x - 19 = 0$.

If we deduce X, X_1, and X_2, we have the three expressions

$$X = 2x^4 - 13x^2 + 10x - 19,$$

$$X_1 = 4x^3 - 13x + 5,$$

$$X_2 = 13x^2 - 15x + 38.$$

If we place $X_2 = 0$, we shall find that both of the roots of the resulting equation are imaginary; hence, X_2 will be positive for all values of x (Art. 325). It is then useless to seek for X_3 and X_4.

By the substitution of $-\infty$ and $+\infty$ in X, X_1, and X_2, we obtain for the first, *two* variations, and for the second *none;* hence, there are two real and two imaginary roots in the proposed equation.

3. Let $x^3 - 5x^2 + 8x - 1 = 0$.

4. $x^4 - x^3 - 3x^2 + x^2 - x - 3 = 0$.

5. $x^5 - 2x^3 + 1 = 0$.

Discuss each of the above equations.