Robinson's Shorter Course.

THE

COMPLETE

ALGEBRA,

DESIGNED FOR USE IN

SCHOOLS, ACADEMIES, AND COLLEGES.

BY

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PREFACE.

THE author of this treatise on Algebra has undertaken the difficult task of preparing a work complete in one volume, which shall be sufficiently thorough for classes in Colleges and Universities, and at the same time sufficiently elementary for classes in Common Schools and Academies. To accomplish this desirable end the work has been so arranged that certain chapters and parts of chapters may be omitted by classes pursuing an elementary course.

The aim has been: 1. To treat each subject in harmony with the present modes of mathematical thinking; 2. To make every statement with such brevity and precision that the student cannot fail to understand the meaning; 3. To give a clear and rigorous demonstration of every proposition; 4. To present *one* difficulty at a time, and just at that stage of the student's progress when he is prepared to understand its treatment; 5. To treat with special care those subjects which have been found by experience to present peculiar difficulties; 6. To make the work thoroughly practical as well as thoroughly theoretical; 7. To present each subject in such a manner as to create a love for the study.

In the arrangement of subjects the author has departed widely from the beaten track; but he feels confident that the plan he has adopted will commend itself to the experienced and thoughtful teacher.

To facilitate frequent reviews, "Synopses for Review" have been placed at convenient intervals throughout the work.

To avoid making the present work too voluminous, Continued Fractions, Reciprocal Equations, Elimination by the Method of the Greatest Common Divisor, and Cardan's formula for cubic equations have been omitted. These subjects are treated in the Appendix to the author's "Book of Algebraic Problems."

In preparing the present treatise the author has first consulted his own experience as a teacher, and the book has been mainly written to meet the wants of his own classes; but he does not hesitate to acknowledge that he has received great assistance from many sources. A part of the material used in the chapters on Positive and Negative Quantities, Greatest Common Divisor and Least Common Multiple, Fractions, Simple Equations, Inequalities, Theory of Exponents, Mathematical Induction, and the sections on Permutations, Combinations, and Logarithms, has been taken from Prof. Todhunter's excellent treatise on Algebra. The works of Bertrand, Young, Peacock, Euler, Bland, Goodwin, and Wrigley have been consulted with advantage.

While the author has availed himself of such material in the books named as suited his purposes, it will be found that much of that so taken has long since become common property, having assumed a stereotyped form; and that other portions have been very much modified. It will be found, also, that the present treatise contains a large amount of new and original matter, which has not been inserted because it was novel, but because it served to simplify and elucidate the subject.

Special attention is called to the full and thorough manner in which the subject of Factoring is treated; to the demonstration of the Lemma, upon which the Binomial Theorem depends; to the classification and treatment of Radical Quantities; to the treatment of Quadratic Equations, Higher Equations, Simultaneous Equations, Ratio, Proportion, Progressions, Interpolation, Recurring Series, Reversion of Series; and to the Theory of Equations.

The chapter on "Logarithms and Exponential Equations" is almost entirely the work of Prof. James M. Greenwood, A. M., Superintendent of the Public Schools of Kansas City, Mo., and formerly Prof. of Math. in the North Missouri State Normal School; and the "Synopses for Review" have nearly all been prepared by Prof. George S. Bryant, A. M., of Christian College, Columbia, Mo. To these and other able and experienced teachers the author is also indebted for many valuable suggestions in relation to other portions of the work.

UNIVERSITY OF THE STATE OF MISSOURI,
COLUMBIA, *January*, 1875.

THE AUTHOR.

SUGGESTIONS TO TEACHERS.

1. If the problems in the book are not sufficiently numerous or sufficiently varied, make some of your own, or take some from the book of "Algebraic Problems," made to accompany this volume.

2. The Synopses for Review should be placed upon the black-board, and dwelt upon until the topics embraced in the review are thoroughly fixed in the mind of the student. To illustrate the manner of conducting a review, suppose the synopsis on page 10 is under consideration. Let the student point to the word "Algebra," and define it; then to "Algebraic Quantity," and define it; then to the two kinds of Algebraic Quantity—"Known and Unknown"—and define them; and so on.

3. The following chapters and parts of chapters may be omitted by classes pursuing an elementary course: That part of Chapter IV from Art. 125 to Art. 128 inclusive, and from Art. 133 to Art. 136 inclusive; Chapter VIII; Chapter XIV; that part of Chapter XVI from Art. 441 to Art. 457 inclusive; Chapter XVII; Arts. 482 and 483 of Chapter XVIII; all of Chapter XX after Geometrical Progressions; Chapter XXI; Chapter XXII; Chapter XXIII.

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2. $x^5 + 5x^4 - 20x^2 - 19x - 2 = 0.$ *Ans.* $\dfrac{2}{2+\sqrt[4]{40}}.$

3. $x^6 - 5x^5 + x^4 + 12x^3 - 12x^2 + 1 = 0.$

4. $x^4 - 8x^3 + 12x^2 + 16x - 39 = 0.$

611. To find the limits of the negative roots of an equation.

Substitute $-x$ for x in the given equation, and find the limits of the positive roots of the resulting equation. By changing the signs of these limits we obtain the limits of the negative roots of the given equation (**602,** Cor. 1).

EXAMPLES.

Find the limits of the negative roots in each of the following equations:

1. $x^3 - 3x^2 + 5x + 7 = 0.$ *Ans.* $-(1 + \sqrt[3]{7}),\ -\dfrac{7}{12}.$

2. $x^4 - 15x^2 - 10x + 24 = 0.$

3. $x^6 - 3x^5 + 2x^4 + 27x^3 - 4x^2 - 1 = 0.$

STURM'S THEOREM.

612. If the coefficients of $f(x)$ are real and the equation $f(x) = 0$ has no equal roots, then, if x is made to assume, in succession, all real values from $-\infty$ to $+\infty$, the sign of $f(x)$ will change as often as x passes a real root of the equation (**608,** Cor. 3). Sturm's Theorem enables us to determine the number of such changes of sign.

613. ***Sturm's Functions.***—Let $f(x) = 0$ be an equation whose coefficients are real, and which is freed from equal roots (**607**); and let $f'(x)$ be the first derivative of $f(x)$.

We now apply to $f(x)$ and $f'(x)$ the process of finding their G. C. D. (**125**), with this modification, namely: 1. *When a remainder is found which is of a lower degree than the corresponding dividend and divisor, we change its sign and use the result for the next divisor.* 2. *We neither introduce nor reject a negative factor in preparing for division.*

We continue the operation until a remainder is obtained which is independent of x, and change the sign of that remainder also.

Let $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$ denote the series of modified remainders thus obtained.

The functions $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$ are called *Sturm's Functions.*

The functions $f'(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$ are called *Auxiliary Functions.*

614. *Sturm's Theorem.*—*If x be conceived to assume, in succession, all real values from $-\infty$ to $+\infty$, there will be no change in the number of variations in the signs of the series of functions $f(x)$, $f'(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, ... $f_n(x)$, except when x passes through a real root of the equation $f(x)=0$; and when x passes through such a root, there will be a loss of only one variation.*

I. $f_n(x)$ is not zero; for, by hypothesis, it is independent of x; hence, if it were zero, $f(x)$ and $f'(x)$ would have a common divisor, and the equation $f(x)=0$ would have equal roots (**607**); but this is contrary to the hypothesis.

II. Two consecutive functions cannot vanish for the same value of x.

Let $q_1, q_2, q_3, \ldots q_n$ denote the successive quotients obtained by performing the operations described in Art. **613**; then, by the principles of division,

$$f(x)=q_1f'(x)-f_1(x) \quad . \; . \; . \quad (1),$$

$$f'(x)=q_2f_1(x)-f_2(x) \quad . \; . \; . \quad (2),$$

$$f_1(x)=q_3f_2(x)-f_3(x) \quad . \; . \; . \quad (3),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{n-2}(x)=q_nf_{n-1}(x)-f_n(x) \quad . \; . \; . \quad (n).$$

Now suppose $f'(x)$ and $f_1(x)$ to vanish at the same time; then by (2) we shall have $f_2(x)=0$; hence by (3), $f_3(x)=0$; and so on; that is, if two consecutive functions vanish at the same time, all the succeeding functions, including $f_n(x)$ would vanish; but this is impossible (I).

III. When any auxiliary function vanishes, the two adjacent functions have contrary signs. Thus, if $f_2(x)=0$, we have by (3), $f_1(x) = -f_3(x)$.

IV. No change can be made in the sign of any one of Sturm's functions, except when x passes through a value which causes that function to vanish (**608**, Cor. 3).

V. Sturm's functions neither gain nor lose a variation of signs when x passes through a value which causes one or more of the auxiliary functions to vanish, but which does not cause $f(x)$ to vanish.

1. Suppose $f_1(x)$ vanishes when $x=c$, and that no other function vanishes for this value of x. Let h be a positive quantity so small that no one of Sturm's functions except $f_1(x)$ vanishes while x is passing from $c-h$ to $c+h$.

When $x=c$, $f'(x)$ and $f_2(x)$ have contrary signs (III); hence they have contrary signs all the time that x is passing from $c-h$ to $c+h$ (IV). Now at the instant x becomes equal to c, $f_1(x)$ changes its sign (**608**, Cor. 3); hence, before the change, its sign is like that of one of the adjacent functions, and after the change it is like that of the other. But no change in the number of variations of signs in a row of signs can be made by simply changing a sign situated between two adjacent contrary signs. Thus, in the row of signs $+-+--+--+-$ there are seven variations; and if we change the fourth sign there are still seven variations.

Hence Sturm's functions neither gain nor lose a variation of signs while x is passing from $c-h$ to $c+h$.

2. Suppose that when $f_1(x)$ vanishes, other auxiliary functions vanish. The vanishing functions cannot be consecutive (II); the functions adjacent to each vanishing function have contrary signs while x is passing from $c-h$ to $c+h$; and each vanishing function changes its sign at the instant x becomes equal to c. But, as we have just shown, this change of sign does not change the number of variations in the row of signs.

VI. Sturm's functions lose one variation of signs, and only one, each time x passes through a real root of the equation $f(x)=0$. Let a be a real root of the equation $f(x)=0$; then $f(a)=0$.

Substituting $a+h$ for x in $f(x)$ and $f'(x)$, and developing by Art. **603**, we have

$$f(a+h)=h\left(f'(a)+f''(a)\frac{h}{\lfloor 2}+f'''(a)\frac{h^2}{\lfloor 3}+\ldots\right) \quad . \quad . \quad . \quad (1),$$

$$f'(a+h)= \quad f'(a)+f''(a)\,h+f'''(a)\frac{h^2}{\lfloor 2}+\ldots \quad . \quad . \quad . \quad (2).$$

Now assume the absolute value of h to be so small that the first term in each of these developments shall be numerically greater than the sum of the other terms; then the sign of $f(a+h)$ will be the same as that of $hf'(a)$, and the sign of $f'(a+h)$ will be the same as that of $f'(a)$. Hence $f(x)$ and $f'(x)$ will have contrary signs when h is negative, and like signs when h is positive. But when h is negative, x is less than a, and when h is positive, x is greater than a; hence when x passes a real root of the equation $f(x)=0$, a variation is changed into a permanence. Now it is evident, from (2), that $f'(x)$ cannot vanish as long as h has such a value that $f'(a)$ is numerically greater than $f''(a)h+f'''(a)\dfrac{h^2}{\lfloor 2}+\ldots$. Some of the auxiliary functions lying between $f'(x)$ and $f_n(x)$ may, however, vanish and change signs while x is passing through the root a; but the change of a sign lying between two adjacent contrary signs (III) does not change the number of variations in the row of signs (V, 1). Therefore, when x passes through the root a, Sturm's functions lose one variation of signs, and only one.

In the same way it may be shown that when x passes through any other real root of the equation $f(x)=0$, Sturm's functions lose another variation of signs.

Cor. 1.—The number of real roots of the equation $f(x)=0$ is equal to the number of variations of signs lost by Sturm's functions while x is passing from $-\infty$ to $+\infty$; the number of real negative roots is equal to the number of variations of signs lost while x is passing from $-\infty$ to 0; and the number of real positive roots is equal to the number of variations of signs lost while x is passing from 0 to $+\infty$.

COR. 2.—Let a be the smallest real root of the equation $f(x) = 0$, b the next greater, c the next, and so on.

Just after x passes through the root a, $f(x)$ and $f'(x)$ have like signs; and just before x passes through the root b, $f(x)$ and $f'(x)$ have contrary signs (VI). But $f(x)$ does not change its sign while x is passing from a to b; hence $f'(x)$ must change its sign. Therefore the equation $f'(x)=0$ has one real root between a and b. In the same way it may be shown that the equation $f'(x) = 0$ has one real root between b and c. Therefore, between any two consecutive real roots of the equation $f(x) = 0$ there is one real root of the equation $f'(x) = 0$.

SCH.—The sign of each remainder is changed in order that there may be neither a gain nor a loss in the number of variations in the row of signs, except when x passes through a real root of the equation (III–V).

EXAMPLES.

Find the number and situation of the real roots of the following equations:

1. $x^3 - 3x^2 - 4x + 13 = 0.$

$$f(x) = x^3 - 3x^2 - 4x + 13,$$
$$f'(x) = 3x^2 - 6x - 4 \quad \textbf{(603)},$$
$$f_1(x) = 2x - 5 \quad \textbf{(613)},$$
$$f_2(x) = +1.$$

ASSUMED VALUES OF x.	FUNCTIONS AND THEIR SIGNS. $f(x)$,	$f'(x)$,	$f_1(x)$,	$f_2(x)$.	NUMBER OF VARIATIONS.
$-\infty$	−	+	−	+	3
0	+	−	−	+	2
1	+	−	−	+	2
2	+	−	−	+	2
3	+	+	+	+	0
$+\infty$	+	+	+	+	0

Hence all the roots of the equation are real; two of them are positive and the other negative; and the two positive roots are situated between 2 and 3.

When $x = -3$ the signs of the functions are $- + - +$, and when $x = -2$ the signs of the functions are $+ + - +$;

hence the negative root is between -2 and -3. To separate the two roots which lie between 2 and 3 we must substitute for x some number or numbers lying between 2 and 3. When $x = 2\frac{1}{2}$ the signs of the functions are $-\ -\ \pm\ +$. Here we have only one variation whether we consider the vanishing function $f_1(x)$ to be positive or negative; hence one of the positive roots lies between 2 and $2\frac{1}{2}$, and the other between $2\frac{1}{2}$ and 3.

2. $x^3 - 3x^2 - 12x + 24 = 0.$

Ans. Three; one between 1 and 2, one between 4 and 5, and one between -3 and -4.

3. $x^3 + 6x^2 + 10x - 1 = 0.$
4. $x^3 - 6x^2 + 8x + 40 = 0.$
5. $x^4 + 4 = 0.$
6. $x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0.$
7. $x^7 - 9x^5 + 6x^4 + 15x^3 - 12x^2 - 7x + 6 = 0.$
8. $x^4 + x^3 - x^2 - 2x + 4 = 0.$

HORNER'S METHOD OF APPROXIMATION.

615. Let it be required to find a root of the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \dots + Kx + L = 0 \quad . \quad . \quad . \quad (1).$$

Suppose a to be the integral part of the root required, and $r, s, t, \dots,$ taken in order, to be the digits of the fractional part.

Let a be found by trial (**608**) or by Sturm's Theorem; then find an equation whose roots shall be less by a than those of (1) (**598**).

Let $y^n + A'y^{n-1} + B'y^{n-2} + \dots + K'y + L' = 0 \dots (2)$ be that equation.

In this equation the value of y is less than 1; hence the terms containing the higher powers of y are comparatively small; neglecting these, we have, approximately,

$$K'y + L' = 0, \quad \text{whence} \quad y = -\frac{L'}{K'}.$$

The first figure in the value of y is r.

Now find an equation whose roots shall be less by r than those of (2). Let $z^n + A''z^{n-1} + B''z^{n-2} + \dots + K''z + L'' = 0 \dots (3)$ be that equation.