Math 194 **Name (Print):** ____________________

Spring 2024

Due in 1 week.

This homework set is a personal questionnaire and some interesting math questions to get you thinking about the concepts that we will see over the course of the semester. The problems are designed to get you thinking, convey a sense of what the class will be like, and to help me get a sense of your background. I am most interested in understanding your background and seeing what math problems get you excited, so on this problem set, it is OK not to completely answer everything. Please explore the problems (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of your guessing, and describe the progress you made. Where were you led by your experimenting?

1. (a) Read the syllabus. Did you find any typos? If you find a typo, then as EXTRA CREDIT I will drop 1 additional homework when computing your final grade.

   (b) What is your math background? What math classes have you previously taken?

   (c) Why are you taking this course?

(d) Do you have any specific goals for this course?

(e) What is your biggest concern about this course?

(f) What is a mathematics topic that you have heard about and would like to learn more about?

2. Write a short description (maybe one paragraph) of how to travel from McHenry Library to your home. Give enough details so that another who student who had your description but no other information can find her/his way. Next class, you will use these written descriptions to give short oral presentations explaining to get to your home.

3. What are matrices? How does one add and multiply them?

4. What are complex numbers? How does one add and multiply them?

5. What are the congruence classes of the integers for the modulus m, i.e. $\mathbb{Z}/m$. How does one add and multiply them?

6. Find the following elements in $\mathbb{Z}/5$: -1, $1/2$, $2/3$, $\sqrt{-1}$. How many of these elements can you find in $\mathbb{Z}/6$?

7. A polynomial $f(x)$ has the *factor-square property* (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not. Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).

   (a) Check that x^2, $x^2 - x$, $x^2 - 2x + 1$, and $x^2 + x + 1$ all have FSP.

   (b) Are x and $x - 1$ the only monic polynomials of degree 1 with FSP?

   (c) Which monic polynomials of degree 2 have FSP?

   (d) Examples of larger degree can be built as products of earlier examples. For instance, the following polynomials of degree 3 all have FSP:

$$\begin{aligned} x^3 + x^2 + x =& x(x^2 + x + 1), \\ x^3 - 1 =& (x - 1)(x^2 + x + 1), \\ x^3 - x^2 =& x^2(x - 1). \end{aligned}$$

   Are there monic FSP polynomials of degree 3 that are not built from FSP polynomials of degree 1 or 2? Similarly, are there examples of degree 4 monic polynomials with FSP that do not arise as products of FSP polynomials of smaller degrees?

   (e) The examples written above all had integer coefficients. Do answers change if we allow polynomials whose coefficients are allowed to be any real numbers? Of if we allow polynomials whose coefficients are complex numbers?

8. A polynomial is *integral* when it has integer coefficients. The square root of 2 is a solution to the integral polynomial equation $x^2 - 2$.

   A number is *rational* when it can be expressed as a/b for integers a and b (with $b \neq 0$).

   A number is *irrational* when it is not rational.

   (a) Suppose that c is a non-square integer. (That is, $c \neq n^2$ for any n.) Explain why $\sqrt{c}$ is not rational. Similarly, if c is a non-cube integer, does it follow that $c^{1/3}$ is irrational?

   (b) Find an integral polynomial equation that has $\alpha = \sqrt{2} + \sqrt{5}$ as a solution. Show that α is irrational.

   (c) Let a and b be integers. Find an integral polynomial equation which has $\sqrt{a} + \sqrt{b}$ as a soluion. Must $\sqrt{a} + \sqrt{b}$ be irrational? Must $\sqrt{a} - \sqrt{b}$ be irrational if $a \neq b$?

   (d) What happens with the number $\beta = \sqrt{3} + \sqrt{5} + \sqrt{7}$? What about $3\sqrt{2} - 2\sqrt{3} - 3\sqrt{5} + \sqrt{6}$ and $5^{1/3} - \sqrt{2}$? Can you formate a general description of what happens.

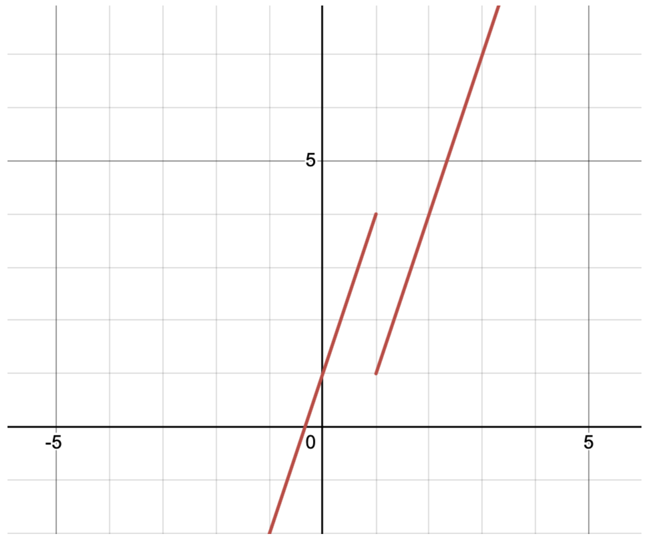

Figure 1: The graph of $f(x)$

9. A professor of experimental physics once asked me, what's the problem with the following mathematical result?

   Theorem: there is function that is differentiable but not continuous.

   Proof: We construct the function explicitly. Let $f(x)\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x - 4 & \text{if } x \geq 1; \\ 3x + 5 & \text{if } x < 1; \end{cases}$$

   Then $f(x)$ is a function that is differentiable but not continuous.

   To see that $f(x)$ is differentiable, we just use the product and addition rules to compute that $f'(x) = 3$.

   We can see from Figure 1 that $f(x)$ is not continuous at $x = 1$. Indeed, $\lim_{x\to 1} f(x) \neq f(1)$ because the limit does not exist. The figure shows (and it is easy to justify algebraically) that the left-hand limit is $+4$ but the right-hand limit is $+1$. Since the one-sided limits are different, the two-sided limit does not exist.

   Certainly this proof can't be correct! What is the problem with it?