MATH 181 NOTES: JANUARY 20, 2022

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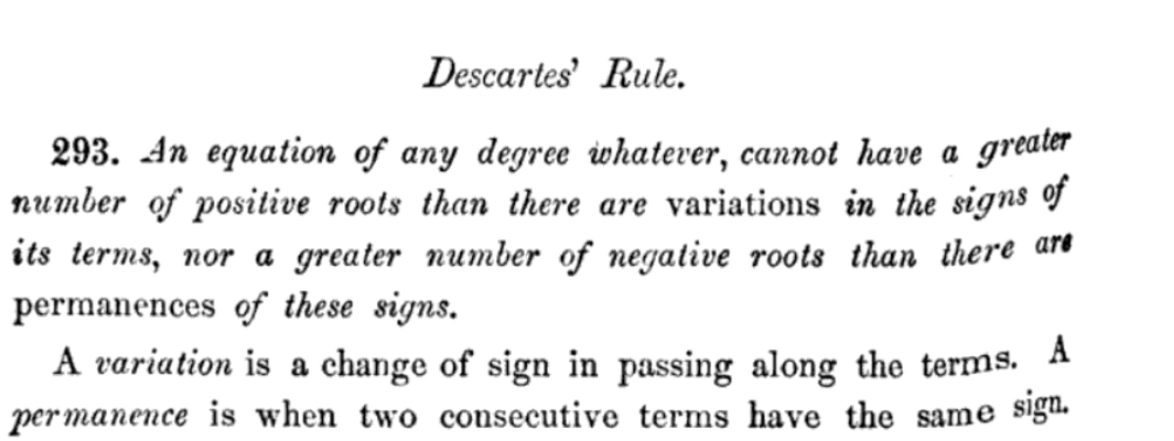
Last class we discussed the biography of T. E. Hart, a professor in nineteenth century America and one of the first Americans to receive a PhD. In this lecture, we will take a closer look at what Hart and other mathematics professors were teaching in the 1850s. At the Citadel Academy, Hart learned college algebra from the textbook *Elements of Algebra on the Basis of M. Bourdon Embracing Sturm's and Horner's Theorems and Practical Examples* by Charles Davies.

In this lecture, the treatment of I will discuss Decartes' Rule of Signs. This is an interesting piece of mathematics that no longer is covered in the college math curriculum. The rule appears in Chapter XI of Davies book. This is the chapter on solving "numerical equations" of one variable (i.e. polynomial equations like $x^2 + 4x + 5$).

The book is organized as a series of numbered "articles." A typical article was begins with a statement of a mathematical rule followed by a description why the rule is true and demonstrating it by examples. Article 277, for instance, explains that, if p and q are numbers such that $f(p)$ and $f(q)$ have opposite signs, then f has a zero that lies between p and q.

Decartes' Rule of Signs appears in Article 293.



Descartes' Rule.

293. ***An equation of any degree whatever, cannot have a greater number of positive roots than there are* variations *in the signs of its terms, nor a greater number of negative roots than there are* permanences *of these signs.***

A *variation* is a change of sign in passing along the terms. A *permanence* is when two consecutive terms have the same sign.

FIGURE 1. Descartes Rule in Davies' textbook

Here's a more modern formulation of the rule:

Theorem 1 (Descartes' rule). *Let*

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

be a polynomial with real coefficients. Let $N(f)$ *equal the number of positive roots (accounting for multiplicity) of* $f(x)$ *and* $V(f)$ *the number of sign changes in the sequence* $1, a_1, a_2, \dots, a_n$ *(ignore the zeros). Then*

$$N(f) \leq V(f).$$

Davies also states a rule involving the negative roots and the number of permanences. I am going to ignore this part because it does not involve any new mathematical issues. The negative roots of $f(x)$ are the same as the positive roots of $f(-x)$, and the number of permanences of $f(x)$ equals the number variations of $f(-x)$, so we get the rule for negative roots by applying the above rule to $f(-x)$.

The rule that I stated is arguable stronger than Davies' rule because it takes into account multiple roots. Davies's somewhat unclear on whether he is counting positive roots or not.

We ended last class by taking a closer look at the mathematics that was taught at Ohio University in the early 1850s. The algebra course was taught from . This was a standard textbook that was used throughout the United States.

As the titled indicates, the textbook is based on French textbook by Louis Pierre Marie (or "M.") Bourdon. In the introduction, Davies refers to this book as "The Treatise on Algebra" which is his translation of the original French title "Élémens d'algèbre." The textbook isn't a direct translation of Bourdon's book. Davies's book is about four hundred pages while some editions of Bourdon's book run to four hundred editions. In the introduction to his book, Davies writes that he used Bourdon's book as a "model," but does not write anything about what specific material he omitted/added and why he made those decisions, and as far as I can tell, nobody else has looked into this issue. It is a natural question to study in the history of math, but it also challenging one as it involved comparing a lengthy textbook written in nineteenth century English with another lengthy textbook written in French.

Davies's introduction provides a little more information. He explains that the textbook developed from his experience teaching from Bourdon's textbook at "the Military Academy." He is referring to the United States Military Academy at West Point. Davies was a student there and then returned as a professor for a number of years. By the 1850s, Davies had stopped teaching and was focusing on writing textbooks full-time. He was very prolific. He wrote over twenty textbooks.

While the presentation and format might be unfamiliar, much of the material should be familiar. The second chapter covers how to do add, subtract, multiply, etc polynomials. However, the later chapters topics you probably haven't seen before like continued fractions and methods for numerically solving polynomial equations like Descartes' rule of signs, Sturm's Theorem, Cardan's Rule, and Horner's Method.

Let's take a close look at Descartes' rule of signs and Sturm's Theorem. We'll start with Descartes' rule since it the most simple of the two. The rule says the following:

Theorem 2 (Descartes' rule). *Let*

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

be a polynomial with real coefficients. Let $N(f)$ *equal the number of positive roots (accounting for multiplicity) of* $f(x)$ *and* $V(f)$ *the number of sign changes in the sequence* $1, a_1, a_2, \ldots, a_n$ *(ignore the zeros). Then*

$$V(f) - N(f) = \text{ a positive even number.}$$

The usefulness of this rule is that it allows you to figure out some information about the roots without doing a lot of work (like factoring the polynomial). The rule isn't widely taught these days, but it remains useful when estimating roots using a computer. Here's an example of how the rule works.

Example 3. Consider the polynomial $f(x) = x^5 - 8x^3 - 2x^2 + 3x + 1$. Descartes' rule tells us to look at the sequence $1, -8, -2, 3, 1$. In this sequence, there are two sign changes (from 1 to -8 and then from -2 to 3. We deduce that $V(f) = 2$ and so the number of positive roots $N(f)$ is a nonnegative number satisfying $N(f) \leq 2$ and $N(f)$ is even. In other words, $N(f) = 0$ or 2.

If we're willing to do use a computer, we can say more about the roots. I had the webapp Demos produce Figure 2. From the graph, it looks like $f(x)$ has 2 roots or, in other words, $N(f) = 2$. (But notice that the negative roots are harder to see; it's unclear if there's two of them or three of them.)

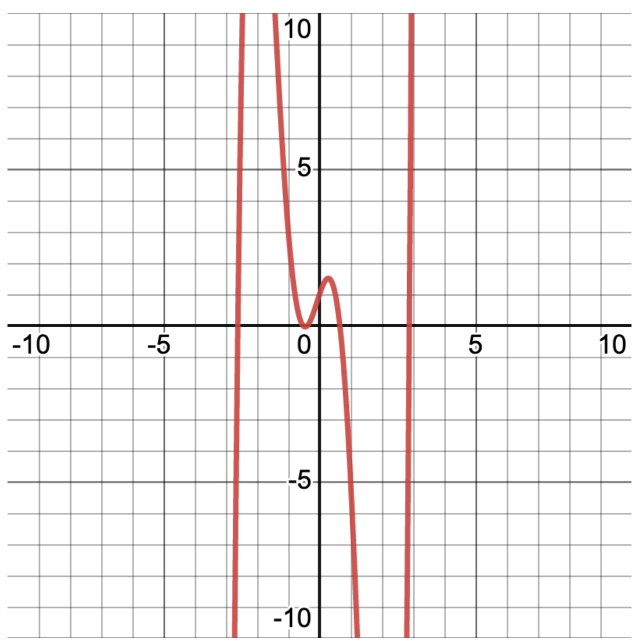


FIGURE 2. The graph of $f(x)$

You can use Wolfram Alpha estimate the roots. The output is in Figure 3. It tells me that there are two positive solutions $r = 2.8794...$ and $r = 0.65270$. Wolfram Alpha doesn't tell you how it performed this computation, but it is probably using methods similar to

Descartes' rule. You should also notice that Wolfram Alpha tells you some *false* information: it tells you that there are two negative solutions that real and all other solutions are given by complex numbers. In fact, all solutions are real.

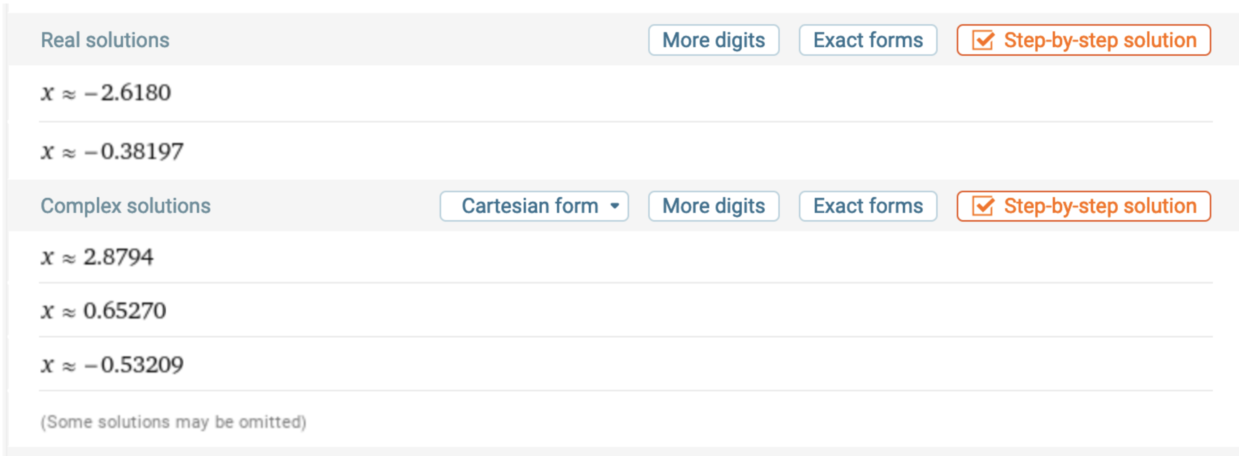


FIGURE 3. Wolfram Alpha's estimates of the roots

Why does Descartes' rule work? Let's take a look at some special cases. When $n = 1$, the polynomial f simplifies to $x + a_1$. Then

$$V(f) = \begin{cases} +1 & \text{if } a_1 < 0; \\ 0 & \text{otherwise.} \end{cases}$$

This equals $Z(f)$ since the only root of f is $x = -a_1$. So we've verified the rule.

What about degree 2. Suppose first that f has two real roots r_1 and r_2 with $r_1 < r_2$. By expanding out $x^2 + a_1x + x_2 = (x - r_1)(x - r_2)$, we get that

$$\begin{aligned} a_1 &= - r_1 - r_2, \\ a_2 &= r_1 r_2. \end{aligned}$$

If $r_1, r_2 < 0$, then the sequence of signs is $+, +, +$, so $V(f) = 0$. This equals $Z(f)$, so we are done in this case.

What about when $r_1, r_2 > 0$? Then the sequence of signs becomes $+, -, +$, so $V(f) = 2$ and we are again done.

The case where $r_1 < 0$ but $r_2 > 0$ is more complicated because the sequence of signs isn't uniquely determined. When $|r_1|$ is larger then $|r_2|$, the sequence of signs is $+, +, -$, but when $|r_2|$ is larger, the sequence is $+, -, -$. However, in both cases, $V(f) = 1$ which equals $Z(f)$.

Another important case to consider is where f has no real roots. By the quadratic formula, this can only happen if $a_1^2 - 4a_2 < 0$ or equivalently $a_2 > a_1^2/4$. In particularly, a_1 can be positive or negative, but we must have that $a_2 > 0$. Thus the sequence of signs is $+, \pm, +$ and $V(f) = 0$ or 2. This is the first case where $V(f)$ does not equal $Z(f)$, but the conclusions in the rule of signs are still satisfied.

We've skipped a few cases, for example where a_1 or a_2 equal zero. Those boundary cases don't involve any new ideas.

The next case is the cubic case, $f(x) = x^3 + a_1x^2 + a_2x + a_3$. When f has three real roots r_1, r_2, r_3, we can take cases as in the quadratic case. A result from calculus is that a polynomial $f(x)$ of degree 3 (or, more generally, odd degree) always has a root. Thus in the case where $f(x)$ does not have three real roots, we can write it as

$$\begin{aligned} f(x) =&(x-r)(x^2+ax+b) \\ =&x^3+(a-r)x^2+(b-ra)-br. \end{aligned}$$

To verify the rule, we need to verify that the number of variations is $V(f) = 1$ or 3 if $r > 0$ and $V(f) = 0$ or 2 if $r < 0$. Again, we analyze what happens on a case-by-case basis. Suppose that $r > 0$ and $|r|$ is much smaller than $|a|$ and $|b$. Then $a - r$ has the same sign as $a - r$ and $b - ra$ has the same sign as b. From our analysis of the quadratic case, we get that the sign sequence is either $+, +, +, -$ or $+, -, +, -$. In other words, the number of variations is either 1 or 3 as the rule claims. Another case to consider if where $r > 0$ and $|r|$ is much larger than $|a|$ and $|b|$, Then the sign sequence is the same as the sign sequence associated to $1, -r, -ra, -br$. This last sequence has sign sequence equal to either $+, -, +, -$ or $+, -, -, -$. Again, we have $V(f) = 1$ or 3, just as the rule predicts. To complete the argument, we need to continue taking cases until we have covered all possibilities.

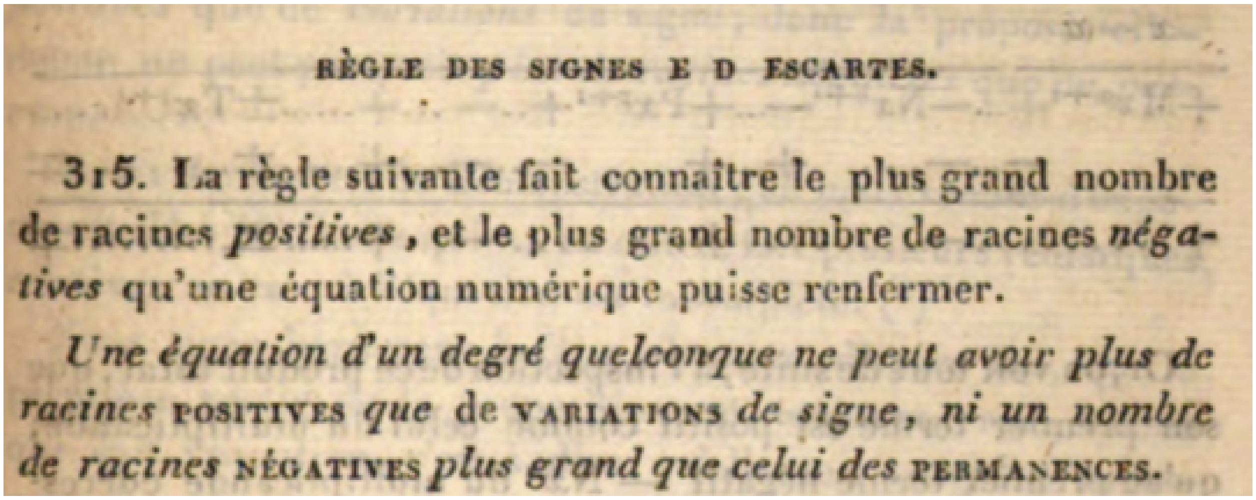
This argument is the contains the essential ideas of the proof in Davies book. Consider verifying Decartes's rule for $(x-r)f(x)$, $r > 0$, when we have already verified the rule for $f(x)$. We have that $N((x-r)f(x)) = 1 + N(f(x))$, so to complete verify the rule, we needed to compute that $V((x-r)f(x)) = 1 + V(f(x))$. This is a lengthy computation in Davies's book.

Davies, in fact, does NOT give a valid proof. (Can you spot the error?) Surprisingly, while Decartes's rule appeared in several English-language textbooks, a complete proof that Decartes's rule was only published in English in 1922 with the book *First Course in the Theory of Equations* by L. E. Dickson.

TRANSMISSION

How was Descartes' rule of signs transmitted to America? Davies tells us that he based his 1837 textbook on *Élémens d'algèbre* by Louis Pierre Marie Bourdon. Figure 4 displays the statement of the rule in Bourdon's textbook. If you can read French or you plug the text in to Google Translate, you'll find that Davies' version of Descartes' rule of signs is a direct translation of the statement from Bourdon. However, this section, and the rest of the textbook overall, is not a direct translation. For example, Davies followed the statement of the rule by an example, while Bourdon jumps directly to a proof.

What was Descartes's role in all of this? He published the rule in *La Géométrie*. This was an appendix to his 1637 book *Discours de la méthode*. *Discours* itself is a work of philosophy, not mathematics. That is where Descartes coined the phrase "I think, therefore I am." *La Géométrie* was written to illustrate general principles for seeking knowledge that



RÈGLE DES SIGNES E D ESCARTES.

315. La règle suivante fait connaître le plus grand nombre de racines *positives*, et le plus grand nombre de racines *négatives* qu'une équation numérique puisse renfermer.

Une équation d'un degré quelconque ne peut avoir plus de racines POSITIVES *que de* VARIATIONS *de signe, ni un nombre de racines* NÉGATIVES *plus grand que celui des* PERMANENCES.

FIGURE 4. Bourdon's statement of Decartes' rule

he laid out in *Discours*. For the same purpose, he also wrote appendices on optics and meteorology.

La Géométrie is a very important book in the history of mathematics. In it, Descartes established analytic geometry, more specifically the idea that plane geometry can be studied by identifying points in the plane with pairs (x, y) of numbers and then working algebraically with those pairs. A 1925 translation of the relevant text D. E. Smith and M. L. Latham is included at the bottom of these notes.

A few things a worth noticing. First, the style is very conversational. Decartes does not break up his text by Definition, Theorem, Proof, Example, etc as was done in the Davies and Bourdon texts. Decartes's rule of signs indeed appears in the text ("An equation can have as many true roots as it contains changes of sign..."), but he doesn't call any particular attention to it even though it is a important, original mathematical result. In fact, Descartes' rule of signs is only the second most important result that Decartes states in that tat selection. He also asserts the Fundamental Theorem of Algebra, the theorem that a polynomial of degree n has n roots over the complex numbers, and he does so in an offhanded manner: "Know then that in every equation there are as many distinct roots as the number of dimensions of the unknown quantity [i.e. the degree of the equation]."

Decartes gives little indication of why the rule of signs is correct, and his statement of the Fundamental Theorem of Algebra is false (not every degree n polynomial has n distinct roots: $(x-1)^2 = x^2 - 2x + 1$ only has one). When reading older mathematical texts, modern readers often assume that the mathematics was more "primitive" in the past: standards of rigor were weaker, peoples understanding of logic was shakier, etc. Decartes appears to know exactly what he is doing. In several places, he explicitly states that he has decided to omit details. The last sentence in the book reads, "I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to other the pleasure of discovery." Decartes quite generally deviated from accepted scholarly practices of his time. He wrote *La Géométrie* in French at a time when the written language of scholars was Latin.

Another thing to notice is that Descartes uses some terminology that seems strange to a modern reader. He says "dimension of an equation" instead of the "degree of a polynomial," and more significantly, refers to negative roots as "false roots." Describing negative numbers as "false" was not uncommon during the early 1600s. Positive numbers can represent observable physical quantities (weight, height, distance,...), while negative numbers do not, and making sense of them requires a greater measure of abstraction (a negative numbers describes a decrease in weight, for example). The terms "false" and "positive" root was used to emphasize this.

On connoiſt auſſy de cecy combien il peut y auoir de vrayes racines, & combien de fauſſes en chaſque Equation. A ſçauoir il y en peut auoir autant de vrayes, que les ſignes + & -- s'y trouuent de fois eſtre changés ; & autant de fauſſes qu'il s'y trouue de fois deux ſignes +, ou deux ſignes -- qui s'entreſuiuent. Comme en la der-

Combien il peut y auoir de vrayes racines en chaſque Equatiō.

FIGURE 5. Descartes' statement of the rule of signs

Of course, Descartes did not write "false root." He wrote "fausses racines" because he was writing in French. This raises another important issue. We need to be very sensitive to word choices, and this presents a particular challenge when reading a foreign language. Figure 5 displays the original text that Descartes wrote, while Figure 6 is the translation by Smith and Latham. This is just one possible way to translate the sentence. When I plugged the text into Google Translate, I go the text in Figure 7. If you compare the original text, you will that Smith and Lantham focused on clarify over preserving Descartes' language. What Descartes wrote as two sentences. In structure, the text Google Translate produced is closer to the original, although the resulting English text is clunky. French and English are pretty similar to each other, and these issues become much more important when we are working with languages that are very different from each other, English and Akkadian for example.

An equation can have as many true roots as it contains changes of sign, from + to — or from — to +; and as many false roots as the number of times two + signs or two — signs are found in succession.

FIGURE 6. Translation of Descartes by Smith and Latham

Similar issues arise with mathematical notation. Take a look at the polynomial equation displayed in Figure 8. The equation is hopefully recognizable as $x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$, but there are a number of differences (how many can you spot?). The most notable is that Descartes does not use the modern equality sign "=" but rather a squiggle that looks somewhat like an ichthus symbol.

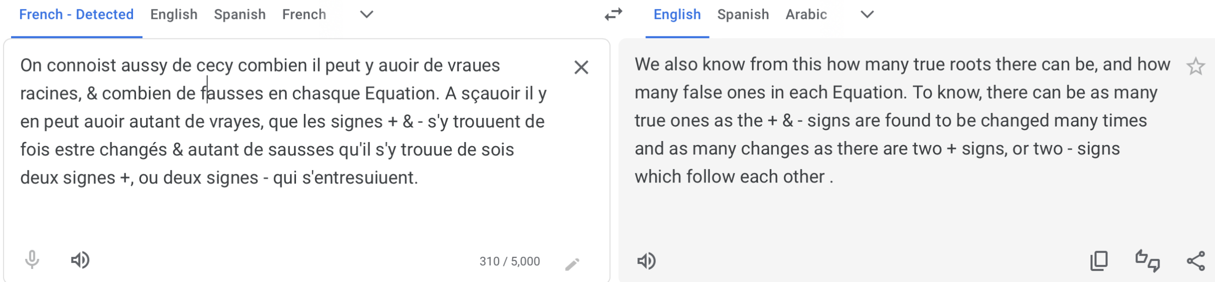


FIGURE 7. Google's translation of Descartes

$$x^4 -- 4x^3 -- 19xx + 106x -- 120 \propto 0$$

FIGURE 8. One of Descartes' equations

There is a lot more to be said about the transmission of the rule of signs over time and space. In a footnote, Smith and Latham explain that Descartes' was not the first person to state the rule of signs in print. The rule appeared five years earlier in Thomas Harriot's book *Artis Analyticae Praxis*. We could take a close look at that text and try to trace the rule of signs back further in time. However, we have already covered more than enough material for one lecture, so I will end here.

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