MATH 181 NOTES: JANUARY 26, 2022

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In our discussion of mathematics in nineteenth century America, we have focused on algebra results, especially Descartes' rule of signs. But a central role – in the nineteenth century and for centuries before that – was played by geometry. The core of geometry was ideas presented by Euclid in his book *Elements*. Last class, I mentioned a quotation by Thomas Jefferson in which he claimed that the alleged inability of Black people to master Euclid as evidence of the racial inferiority of "the blacks." Despite Jefferson's comments, Black Americans did, of course, learn geometry. As we saw in class, J. C. Corbin published several problems and solutions to geometry problems. Daniel Payne, an influential bishop in the African Methodist Episcopalian Church, recalled studying the first five books of *Elements* in his autobiography.

Jefferson was not the only US president to have written about the importance of Euclid. As an adult, Abraham Lincoln devoted a great deal of time to studying the first six books of Euclid. One acquaintance recalled Lincoln saying that

> At last I said [to myself], "Lincoln, you can never make a lawyer if you do not understand what demonstrate means"; and I left my situation in Springfield[, Illinois where he was working as a lawyer], went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what "demonstrate" means, and went back to my law-studies.

Lincoln scholars believe parts of this recollection are false, but by many accounts, Lincoln closely studied Euclid. Moreover, Jefferson and Lincoln were not the only US presidents who closely studied him. President Garfield published his own proof of the Pythagorean theorem. And they were part of a two-thousand year tradition. Euclid's work was a cornerstone of education in the west until the late twentieth century.

Despite the significance of Euclid's work, we know frustratingly little about him. The most reliable source of information about him is the following passage by the Greek philosopher Proclus:

> Not much younger than these (sc. Hermotimus of Colophon and Philippus of Medma) is Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived in the time of the first Ptolmy. For Archimedes, who came immediately after the first (Ptolmy), makes mention of Euclid: and, further, they say that Ptolmey once asked him if there was in geometry any shorter way than that of the elements, and he

> answered that there was no royal road to geometry. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says.

We can pin down the approximate period during which Euclid lived and worked based on the other people Proclus mentions. The "Ptolmy" that is referenced is Ptolemy I Soter, a king who ruled a region based in modern Egypt. (Of course, Egypt is not in Greece, but the region, and much of the area around the Mediterranean, was part of Greek culture.) Ptolemy ruled from 306 to 283 BC. Proclus also mentioned Plato, Eratosthenes, and Archimedes. Plato died in 347/8 BC, while Eratosthenes and Archimedes were born circa 280 BC. This indicates that Euclid did his work around 300 BC.

As is clear from the passage, Proclus had no direct knowledge of Euclid. Indeed, the two are separated by a vast gulf of time. Proclus was born in 412 AD, roughly seven-hundred years after Euclid. This suggests that we view Proclus's passage critically (think about how little we know about people who lived in 1300 AD).

Many scholars believe that Euclid lived in the city of Alexandria, part of Ptolmy I's kingdom. As with all other personal details about Euclid, the evidence for this is frustratingly thin. The main evidence for this is a remark by the Greek mathematician Pappus that another Greek mathematician (Appollonius) had studied geometry under pupils of Euclid at Alexandria. Make of this what you will. Pappus does not explicitly say that Euclid was lived in Alexandria (certainly his pupils could have moved there from a different city), and in any case, Pappus was writing in the early AD 400s, so he was writing about a distant past.

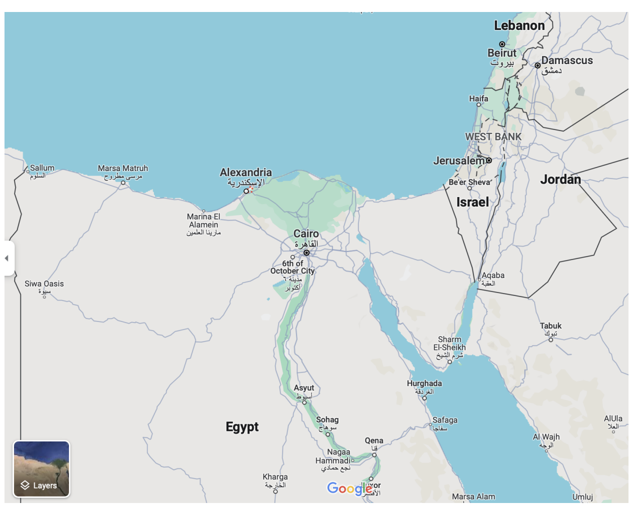

FIGURE 1. The location of Alexandria

There are bits and pieces of information about Euclid in other sources, but everything was written hundreds of years after he died and is of questionable accuracy. One of the better stories is the following one, recounted by the Greek writer Joannes Stobaeus:

> some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, 'But what shall I get by learning these things?" Euclid called his slave and said 'Give him threepence, since he must make gain out of what he learns'.

Accurate or not, the story is perhaps a good one to pull out if you have relatives who give you a hard time about studying "impractical" subjects in college.

Our knowledge of the book *Elements* is similar to that of its author. The earliest text we have is displayed in Figure 2. It isn't much! A single torn papyrus that was found in an Egyptian garbage dump in 1897. Based on the style of writing, the classicist Eric Gardner Turner dates the fragment to circa AD 75–125, perhaps four hundred years after *Elements* was first written.

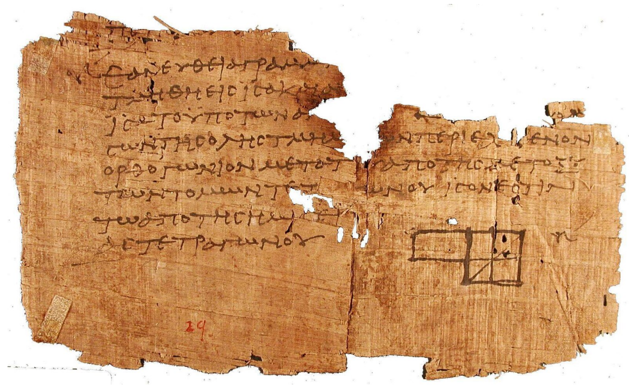

FIGURE 2. A papyrus fragment of Euclid's *Elements*

The earliest full copy of *Elements* is a volume from the 9th-century AD that is currently held by the Vatican. The book has been digitized, and a page from it is displayed in Figure 3.

Perhaps what is most notable of Euclid's *Elements* is that it has been continuously preserved for over two millennia. This is a remarkably cultural achievement. There is nothing similar for many other cultures. By comparison, nobody could read any texts from ancient Mesopotamia until the 1850s when scholars figured out how to decipher the language.

Nevertheless, recovering what Euclid originally wrote is a major challenge. The texts we have are copies of copes of copies of..., and each time the text was copied (often by hand), the writer may have made changes. Thankfully for us, this is a challenge that has largely been overcome. In the nineteenth century, classicists, most notably Johan Ludvig Heiberg and Thomas Little Heath, spent a great deal of time comparing the extend copies of *Elements* in order to produce a critical edition that was as close to what Euclid wrote as possible. Their work has held up well, so unless a new significant copy of *Elements* is found, their published editions are likely the best that can be produced. Heath also translated the text into English, and his English version remains the definitive translation.

FIGURE 3. The Vatican's copy of *Elements*

So what is in *Elements*. Here's how the text begins:

> Definitions.
> (1) A point is that which has no part.
> (2) A line is breadthless length.

Euclid proceeds to give twenty-one more definitions. He then gives five postulates concerning the geometry of the plane.

> Postulates. Let the following be postulated:
> (1) To draw a straight line from any point to any point.
> (2) To produce a finite straight line continuously in a straight line.
> (3) To describe a circle with any centre and distance.
> (4) That all right angles are equal to one another.
> (5) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

These postulates are complemented by five common notions:

> Common notions.

(1) Things which are equal to the same thing are also equal to one another.
(2) If equals be added to equals, the wholes are equal.
(3) If equals be subtracted from equals, the remainders are equal.
(4) Things which coincide with one another equal to one another.
(5) The whole is greater than the part.

Euclid then states his first proposition and proceeds to prove it using the postulates and common notions.

Despite being the most widely read mathematical text, *Elements* is hardly inviting. Euclid evidently assumes that his readers have had considerable exposure to theoretical mathematics. This raises the question, what was Greek mathematics like before Euclid? Euclid's style of writing strongly suggests that it was a mature body of knowledge but like so much else with early Greek mathematics, we unfortunately have very limited evidence to work with. In terms of textual evidence, all that we have are a few scraps of mathematical texts as well as references to mathematics in otherwise non-mathematical texts like works of philosophy.

One of the most important texts is the Socratic dialogue *Meno*, written by Plato. *Meno* was written around 380 BC, about eighty years before *Elements*, so it gives us important insight into mathematics before Euclid. The text was written to explain Plato's ideas about virtue (whether it is in-born or learned), but it includes a discussion of mathematics. The discussion occurs during a debate between the philosopher Socrates and the Greek general Meno. To demonstrate a point, Socrates asks a young man enslaved by Meno a math question: starting with a square of area 4, how to construct a square with area 8? The young man initially gives the wrong answer, but after responding to questions by Socrates, he ultimately gives the correct answer. What is significant for our class is that this discussion demonstrates that Greek mathematics was already quite advanced and many educated people had some knowledge of it.

How did this sophisticated Greek mathematics develop? Before the 1970s, there was widely accepted theory that ran as follows. At some point between 580 and 500 BC, the Greek philosopher Pythagoras visited Egypt and Babylon and learned of the mathematics being done there. He returned to Greece and started a sort of religious cult in which mathematics played a central role. A key doctrine promoted by Pythagoras was that "all is number." As part of their religious work, Pythagoras and his followers disseminated and then further developed Babylonian and Egyptian mathematics. One story holds that the Pythagorian's religious beliefs were challenged by the discovery that $\sqrt{2}$ is irrational, not a "number" in the sense the Pythagoreans understood. The mathematics result, it was said, was a closely guarded secret, and to prevent it from being divulged, the Pythagoreans even went so far as to drown a follower who shared it with non-believers.

Following scholarly work by German classicist Walter Burkert, few scholars believe this theory any more. The evidence just isn't there. All accounts of Pythagoras doing mathematics were written hundreds of years after he died, and a skeptical reading suggests that they have no basis in fact. An alternative theory that has been proposed is that the Greek mathematics of Euclid developed out of practical work done by craftsmen (surveys, accountants,...) when Greek philosophers became interested in the theoretical basis of this work. There was a sharp social divide between philosophers (elite members of society) and craftsmen, so people like Euclid took care to deemphasize the practical origins of mathematics. Ultimately, our sources of information about Greek mathematics before Euclid are limited, and absent a major scholarly breakthrough, there is little we can say with certainty.

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This is part of the text of the Socratic dialogue “Meno” by Plato. The translation by Benjamin Jowett, avilalbe on the website http://classics.mit.edu/Plato/meno.html.

Men. Yes, Socrates; but what do you mean by saying that we do not learn, and that what we call learning is only a process of recollection? Can you teach me how this is?

Soc. I told you, Meno, just now that you were a rogue, and now you ask whether I can teach you, when I am saying that there is no teaching, but only recollection; and thus you imagine that you will involve me in a contradiction.

Men. Indeed, Socrates, I protest that I had no such intention. I only asked the question from habit; but if you can prove to me that what you say is true, I wish that you would.

Soc. It will be no easy matter, but I will try to please you to the utmost of my power. Suppose that you call one of your numerous attendants, that I may demonstrate on him.

Men. Certainly. Come hither, boy.

Soc. He is Greek, and speaks Greek, does he not?

Men. Yes, indeed; he was born in the house.

Soc. Attend now to the questions which I ask him, and observe whether he learns of me or only remembers.

Men. I will.

Soc. Tell me, boy, do you know that a figure like this is a square?

Boy. I do.

Soc. And you know that a square figure has these four lines equal?

Boy. Certainly.

Soc. And these lines which I have drawn through the middle of the square are also equal?

Boy. Yes.

Soc. A square may be of any size?

Boy. Certainly.

Soc. And if one side of the figure be of two feet, and the other side be of two feet, how much will the whole be? Let me explain: if in one direction the space was of two feet, and in other direction of one foot, the whole would be of two feet taken once?

Boy. Yes.

Soc. But since this side is also of two feet, there are twice two feet?

Boy. There are.

Soc. Then the square is of twice two feet?

Boy. Yes.

Soc. And how many are twice two feet? count and tell me.

Boy. Four, Socrates.

Soc. And might there not be another square twice as large as this, and having like this the lines equal?

Boy. Yes.

Soc. And of how many feet will that be?

Boy. Of eight feet.

Soc. And now try and tell me the length of the line which forms the side of that double square: this is two feet-what will that be?

Boy. Clearly, Socrates, it will be double.

Soc. Do you observe, Meno, that I am not teaching the boy anything, but only asking him questions; and now he fancies that he knows how long a line is necessary in order to produce a figure of eight squarefeet; does he not?

Men. Yes.

Soc. And does he really know?

Men. Certainly not.

Soc. He only guesses that because the square is double, the line is double.

Men. True.

Soc. Observe him while he recalls the steps in regular order. (To the Boy.) Tell me, boy, do you assert that a double space comes from a double line? Remember that I am not speaking of an oblong, but of a figure equal every way, and twice the size of this-that is to say of eight feet; and I want to know whether you still say that a double square comes from double line?

Boy. Yes.

Soc. But does not this line become doubled if we add another such line here?

Boy. Certainly.

Soc. And four such lines will make a space containing eight feet?

Boy. Yes.

Soc. Let us describe such a figure: Would you not say that this is the figure of eight feet?

Boy. Yes.

Soc. And are there not these four divisions in the figure, each of which is equal to the figure of four feet?

Boy. True.

Soc. And is not that four times four?

Boy. Certainly.

Soc. And four times is not double?

Boy. No, indeed.

Soc. But how much?

Boy. Four times as much.

Soc. Therefore the double line, boy, has given a space, not twice, but four times as much.

Boy. True.

Soc. Four times four are sixteen-are they not?

Boy. Yes.

Soc. What line would give you a space of right feet, as this gives one of sixteen feet;-do you see?

Boy. Yes.

Soc. And the space of four feet is made from this half line?

Boy. Yes.

Soc. Good; and is not a space of eight feet twice the size of this, and half the size of the other?

Boy. Certainly.

Soc. Such a space, then, will be made out of a line greater than this one, and less than that one?

Boy. Yes; I think so.

Soc. Very good; I like to hear you say what you think. And now tell me, is not this a line of two feet and that of four?

Boy. Yes.

Soc. Then the line which forms the side of eight feet ought to be more than this line of two feet, and less than the other of four feet?

Boy. It ought.

Soc. Try and see if you can tell me how much it will be.

Boy. Three feet.

Soc. Then if we add a half to this line of two, that will be the line of three. Here are two and there is one; and on the other side, here are two also and there is one: and that makes the figure of which you speak?

Boy. Yes.

Soc. But if there are three feet this way and three feet that way, the whole space will be three times three feet?

Boy. That is evident.

Soc. And how much are three times three feet?

Boy. Nine.

Soc. And how much is the double of four?

Boy. Eight.

Soc. Then the figure of eight is not made out of a of three?

Boy. No.

Soc. But from what line?-tell me exactly; and if you would rather not reckon, try and show me the line.

Boy. Indeed, Socrates, I do not know.

Soc. Do you see, Meno, what advances he has made in his power of recollection? He did not know at first, and he does not know now, what is the side of a figure of eight feet: but then he thought that he knew, and answered confidently as if he knew, and had no difficulty; now he has a difficulty, and neither knows nor fancies that he knows.

Men. True.

Soc. Is he not better off in knowing his ignorance?

Men. I think that he is.

Soc. If we have made him doubt, and given him the "torpedo's shock," have we done him any harm?

Men. I think not.

Soc. We have certainly, as would seem, assisted him in some degree to the discovery of the truth; and now he will wish to remedy his ignorance, but then he would have been ready to tell all the world again and again that the double space should have a double side.

Men. True.

Soc. But do you suppose that he would ever have enquired into or learned what he fancied that he knew, though he was really ignorant of it, until he had fallen into perplexity under the idea that he did not know, and had desired to know?

Men. I think not, Socrates.

Soc. Then he was the better for the torpedo's touch?

Men. I think so.

Soc. Mark now the farther development. I shall only ask him, and not teach him, and he shall share the enquiry with me: and do you watch and see if you find me telling or explaining anything to him, instead of eliciting his opinion. Tell me, boy, is not this a square of four feet which I have drawn?

Boy. Yes.

Soc. And now I add another square equal to the former one?

Boy. Yes.

Soc. And a third, which is equal to either of them?

Boy. Yes.

Soc. Suppose that we fill up the vacant corner?

Boy. Very good.

Soc. Here, then, there are four equal spaces?

Boy. Yes.

Soc. And how many times larger is this space than this other?

Boy. Four times.

Soc. But it ought to have been twice only, as you will remember.

Boy. True.

Soc. And does not this line, reaching from corner to corner, bisect each of these spaces?

Boy. Yes.

Soc. And are there not here four equal lines which contain this space?

Boy. There are.

Soc. Look and see how much this space is.

Boy. I do not understand.

Soc. Has not each interior line cut off half of the four spaces?

Boy. Yes.

Soc. And how many spaces are there in this section?

Boy. Four.

Soc. And how many in this?

Boy. Two.

Soc. And four is how many times two?

Boy. Twice.

Soc. And this space is of how many feet?

Boy. Of eight feet.

Soc. And from what line do you get this figure?

Boy. From this.

Soc. That is, from the line which extends from corner to corner of the figure of four feet?

Boy. Yes.

Soc. And that is the line which the learned call the diagonal. And if this is the proper name, then you, Meno's slave, are prepared to affirm that the double space is the square of the diagonal?

Boy. Certainly, Socrates.

Soc. What do you say of him, Meno? Were not all these answers given out of his own head?

Men. Yes, they were all his own.

Soc. And yet, as we were just now saying, he did not know?

Men. True.

Soc. But still he had in him those notions of his-had he not?

Men. Yes.

Soc. Then he who does not know may still have true notions of that which he does not know?

Men. He has.

Soc. And at present these notions have just been stirred up in him, as in a dream; but if he were frequently asked the same questions, in different forms, he would know as well as any one at last?

Men. I dare say.

Soc. Without any one teaching him he will recover his knowledge for himself, if he is only asked questions?

Men. Yes.

Soc. And this spontaneous recovery of knowledge in him is recollection?

Men. True.

Soc. And this knowledge which he now has must he not either have acquired or always possessed?

Men. Yes.

Soc. But if he always possessed this knowledge he would always have known; or if he has acquired the knowledge he could not have acquired it in this life, unless he has been taught geometry; for he may be made to do the same with all geometry and every other branch of knowledge. Now, has any one ever taught him all this? You must know about him, if, as you say, he was born and bred in your house.

Men. And I am certain that no one ever did teach him.

Soc. And yet he has the knowledge?

Men. The fact, Socrates, is undeniable.