MODULI OF GENERALIZED DIVISORS ON THE RULED CUBIC SURFACE

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ABSTRACT. Here we describe the moduli space of generalized divisors on the ruled cubic surface.

In this note we describe the moduli space of generalized divisors, or rank 1 reflexive sheaves, on the ruled cubic surface X. Examples of generalized divisors are line bundles. These do not have interesting moduli: they are rigid, so the moduli space of line bundles is just a discrete set of points. The geometry becomes more interesting if we allow sheaves that fail to be locally free. For example, the elements of the ruling of the surface define a 1-dimensional family of rank 1 reflexive sheaves. In fact, we will see that this family contains sheaves that are embedded points of the moduli space.

Recall that the ruled cubic surface is a nonnormal surface projective space whose normalization is the blow-up $\widetilde{X} := \mathrm{Bl}_{p_0}(\mathbf{P}^2_k)$ of the plane at a point. The linear system of quadratics passing through p_0 embeds $\widetilde{X}$ in $\mathbf{P}^4$, and X is the image of $\widetilde{X}$ under a general projection $\pi\colon \mathbf{P}^4_k \dashrightarrow \mathbf{P}^3_k$.

A simplified form of the main theorem is the following:

Theorem 1. *Let $\mathcal{O}(D)$ be a reflexive rank 1 sheaf on X. If $\mathcal{O}(D)$ fails to be locally free along a curve, then $\{\mathcal{O}(D)\}$ is a connected component of the moduli space $\overline{M}(X)$ of reflexive rank 1 sheaves.*

Otherwise, $\mathcal{O}(D)$ lies on an irreducible component of dimension $\geq r$, where $r < \infty$ is the number of closed points where $\mathcal{O}(D)$ fails to be locally free.

To state the full result, we need to introduce some notation for rank 1 reflexive sheaves. The surface X fails to be normal along the curve defined by the conductor ideal. We denote this curve by D_{sing}. Write $\widetilde{D}_{\text{sing}} \subset \widetilde{X}$ for its preimage under the normalization map. The restriction $\widetilde{D}_{\text{sing}} \to D_{\text{sing}}$ of the normalization map is a double cover of a rational curve by a rational curve. The map ramified at two points that we call the **pinch points** of X.

In [Har94], Hartshorne describes the rank 1 reflexive sheaves on X as follows. Sheaves that fail to be locally free at a finite set of points form a group: the almost Picard group $\mathrm{APic}(X)$. Hartshorne constructs a homomorphism

$$\phi = \phi_1 \times \phi_2\colon \mathrm{APic}(X) \to \mathbf{Z}^2 \times \mathrm{Div}(\widetilde{D}_{\text{sing}})/\pi^*\mathrm{Div}(D_{\text{sing}})$$

that is injective with image equal to the set of pairs $(a, b; [\alpha])$ such that $a = \deg(\alpha)$ mod 2.

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The set $\mathrm{GPic}(X)$ of all rank 1 reflexive sheaves can be described in terms of $\mathrm{APic}(X)$. The sheaf $\mathcal{O}(D_{\mathrm{sing}})$ is an element of $\mathrm{GPic}(X) - \mathrm{APic}(X)$, and every element of this set can be written as $\mathcal{O}(C + D_{\mathrm{sing}})$ for some $C \in \mathrm{APic}(X)$. Furthermore, $\mathcal{O}(C_1 + D_{\mathrm{sing}}) = \mathcal{O}(C_2 + D_{\mathrm{sing}})$ if and only if $\mathcal{O}(C_1)$ and $\mathcal{O}(C_2)$ have the same image in $\mathbf{Z}^2$. With this notation, we can state the main result more precisely:

Definition 2. For $\mathcal{O}(D) \in \mathrm{APic}(X)$, let $h(\mathcal{O}(D))$ denote the minimal degree of an effective divisor that represents $\phi_2(\mathcal{O}(D)) \in \mathrm{Div}(\widetilde{D}_{\mathrm{sing}}/\pi^*\mathrm{Div}(D_{\mathrm{sing}}))$.

Theorem 3. *If $\mathcal{O}(D) \in \mathrm{GPic}(X) - \mathrm{APic}(X)$, then $\{\mathcal{O}(D)\}$ is a connected component of $\overline{M}(X)$.*

Given $a, b, n \in \mathbf{Z}$ with $a = n \bmod 2$, the subset of sheaves $\mathcal{O}(D)$ with $\phi_1(\mathcal{O}(D)) = (a, b)$ and $h(\mathcal{O}(D)) = n$ is an irreducible component of dimension n.

We also prove results about the nonreducedness of $\overline{M}(X)$.

Theorem 4. *The moduli space $\overline{M}(X)$ is nonreduced at sheaves that fail to be locally free along a curve.*

If $\mathcal{O}(D)$ satisfies $\phi(\mathcal{O}(D)) = (1, 1; [\text{Pinch Point}])$, then $\mathcal{O}(D)$ is an embedded point of $\overline{M}(X)$.

Using an Abel map, one can deduce as a corollary analogous results about the Hilbert scheme $H_{d,g}$ of degree d, genus g Cohen-Macaulay curves on X.

I. **Comparison with past work.** The moduli space of rank 1, torsion-free sheaves on an irreducible projective variety X was constructed by Altman–Kleiman in [AK79, AK80]. When X is a curve, there is a large volume of results on the geometry of the moduli space. For example, the articles [Reg80, AIK77, KK81] prove that the line bundle locus is dense if and only if X has at worst plane curve singularities.

When the dimension of X is two or more, there are very few results about the geometry of $\overline{M}(X)$. When X is normal, by [Kle05, Theorem 5.4], the line bundle locus is closed and hence a union of connected component. The construction in [AK75] shows that this is not true for the larger locus of reflexive sheaves; when X is the cone over a plane cubic (a normal surface in $\mathbf{P}^3_k$), the closure of this locus contains rank 1, torsion-free sheaves which are not reflexive.

The strongest positive results hold when X is a connected component of the moduli space $\overline{M}^d(C)$ of rank 1, torsion-free sheaves on an integral curve C/k. In this case, the main result of [Ari13, Theorems B] (extending results from [EK05]) states that the connected component of $\overline{M}(X)$ containing $\mathcal{O}_X$ is isomorphic to X itself. The result is proven by exhibiting an explicit isomorphism, and the construction shows that all elements in the connected component are Cohen-Maculay (and hence reflexive) sheaves [Ari13, Theorems A].

Background. Here we collect some basic results from the literature. We introduced the ruled cubic surface as a birational model of the blow up of the plane, but for later computations, it is useful to have an explicit model. We take X to be defined by the equation

$f = W^2X - Y^2Z$, i.e. $X = \mathrm{Proj}\, k[W, X, Y, Z]/f$. With $\widetilde{X} := \mathrm{Bl}_{p_0}(\mathbf{P}^2_k)$ and $p_0 := [0, 0, 1]$, the normalization map $\pi\colon \widetilde{X} \to X$ is the morphism induced by the morphism $\mathbf{P}^2_k \to X$ defined in projective coordinates by $\pi(S, T, U) = [ST, U^2, SU, T^2]$. In terms of this projective model, the singular locus D_{sing} is the line $\{W = Y = 0\}$. Set $\widetilde{D}_{\mathrm{sing}} \subset \widetilde{X}$ equal to the preimage of D_{sing}. The curve $\widetilde{D}_{\mathrm{sing}}$ is the total transform of the curve $\{S = 0\} \subset \mathbf{P}^2_k$, and $\pi\colon \widetilde{D}_{\mathrm{sing}} \to D_{\mathrm{sing}}$ is a double cover. The ramification points are the points $[0, 1, 0, 0]$ and $[0, 0, 0, 1]$. We call these points the pinch points of X. By abuse of language, we also call their preimages pinch points.

The cohomology of X is

$$h^i(X, \mathcal{O}) = \begin{cases} 1 & \text{if } i = 0; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

To see, use the analogous computation of $h^i(\widetilde{X}, \mathcal{O})$ [Har77, Corollary 2.5] and the short exact sequence

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_{\widetilde{X}} \to \pi_* \mathcal{O}_{\widetilde{D}_{\mathrm{sing}}} / \mathcal{O}_{D_{\mathrm{sing}}} \to 0.$$

(Use [HP15, Remark 2.7] to show that $\pi_* \mathcal{O}_{\widetilde{D}_{\mathrm{sing}}} / \mathcal{O}_{D_{\mathrm{sing}}}$ is indeed the cokernel.)

We now define some divisors that will play an important role. The image $\widetilde{E} \subset \widetilde{X}$ of the exceptional divisor is the line $E := \{X = Z = 0\}$. An another important class of divisors are the rulings of X. Given a closed point $p \in E$, write L_p for the unique element of the ruling that passes through p. Concretely, if $p = [b, 0, a, 0]$, then $L_p = \{bY = aW, b^2X = a^2Z\}$. A third example of a divisor is a hyperplane section H. By the adjunction formula, the dualizing sheaf ω_X of X is isomorphic to $\mathcal{O}(-H)$.

As mentioned in the introduction, Hartshorne constructed an injection $\mathrm{APic}(X) \to \mathbf{Z}^2 \times \mathrm{Div}(\widetilde{D}_{\mathrm{sing}})/\pi^*\mathrm{Div}(D_{\mathrm{sing}})$ with image equal to the elements $(a, b; [\alpha])$ satisfying $a = \deg(\alpha) \bmod 2$. The key property of this map is the following one: if D is an effective divisor that does not contain D_{sing}, then let $\widetilde{D}$ denote the Zariski closure of $\pi^{-1}(D - D \cap D_{\mathrm{sing}})$. The integers $(a, b) := \phi_1(\mathcal{O}(D))$ are the unique integers satisfying $\mathcal{O}(\widetilde{D}) = \mathcal{O}(a\widetilde{L} - b\widetilde{E})$ for $\widetilde{E}, \widetilde{L} \subset \widetilde{X}$ equal to the exceptional divisor and the total transform of a line respectively. The class $[\alpha]$ is the image of the divisor $\widetilde{D} \cap \widetilde{D}_{\mathrm{sing}}$. With this description, we get that

$$\begin{aligned} \phi(\mathcal{O}(E)) =& (0, -1; 0), \\ \phi(\mathcal{O}(L_p)) =& (1, 1; [p]), \\ \phi(\mathcal{O}(H)) =& (2, 1; 0). \end{aligned}$$

We now turn our attention to the moduli space $\overline{M}(X)$. This scheme represents the étale sheaf associated to the functor that sends a k-scheme T to the set of isomorphism classes of flat families of rank 1, torsion-free sheaves parameterized by T. By [AK80, (3.1) Theorem], this scheme exists and its connected components are proper. The reflexive sheaves form an open subscheme by [AK79, (5.13) Proposition]. (Note: the statement of the proposition states that the Cohen–Macaulay (or "pseudo-invertible") sheaves form an open subscheme, but because X is a surface, [Har94, Corollary 1.8] implies that reflexivity is equivalent to Cohen–Macaulayness)

A important tool in studying $\overline{M}(X)$ is the Abel map from the Hilbert scheme to $\overline{M}(X)$. The Hilbert scheme (parameterizing subschemes of X) exists [AK79, (2.8) Corollary] as a scheme. Furthermore, if we fix the Hilbert polynomial with respect to the ample line bundle $\mathcal{O}(D)$, then the corresponding locus in the Hilbert scheme is a closed and open subsubscheme that is projective. Let $H(X) \subset \mathrm{Hilb}(X)$ denote the open subscheme parameterizing subschemes that are pure of dimension 1 with no embedded points or, equivalently [Har94, Proposition 2.4], have reflexive ideal sheaf. Given such a subscheme $D \subset X$, the ideal sheaf $\mathcal{O}(-D)$ is Cohen–Macaulay and thus $\mathit{Ext}^1(\mathcal{O}(-D), \mathcal{O})$ vanishes. We conclude by [AK80, 1.10 Theorem] that the formation of $\mathcal{O}(D)$ behaves well in families and thus defines a morphism $\mathrm{Abel}\colon H(X) \to \overline{M}(X)$. The fibers of Abel are the described by the following lemma which is essentially [AK80, (5.18) Theorem].

Lemma 5. *The fiber of* $\mathrm{Abel}\colon H(X) \to \overline{M}(X)$ *over* $\mathcal{O}(D)$ *is the projective space* $\mathbf{P}H^0(X, \mathcal{O}(D))$ *of one-dimensional subspaces of* $H^0(X, \mathcal{O}(D))$. *If* $H^1(X, \mathcal{O}(D)) = 0$, *then* Abel *is smooth along the fiber over* $\mathcal{O}(D)$.

Proof. The morphism Abel is the composition of the morphism $D \mapsto \mathcal{O}(-D)$ followed by the involution $\mathcal{O}(D) \mapsto \mathit{Hom}(\mathcal{O}(D), \mathcal{O})$. By [AK80, (1.1.1), (5.17(i))], the fiber of Abel over $\mathcal{O}(D)$ is $\mathbf{P}\mathrm{Hom}(\mathcal{O}(-D), \mathcal{O})$ which equals $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(D)) = H^0(X, \mathcal{O}(D))$ by reflexivity.

To prove the second statement, consider the local-to-global spectral sequence $E^2_{p,q} = H^p(\mathit{Ext}^q(\mathcal{O}(-D), \mathcal{O})) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{O}(-D), \mathcal{O})$. Because $\mathcal{O}(-D)$ is reflexive (and hence is Cohen–Macaulay), we have $\mathit{Ext}^1(\mathcal{O}(-D), \mathcal{O}) = 0$ and thus

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}(-D), \mathcal{O}) =& H^1(\mathit{Ext}^0(\mathcal{O}(-D), \mathcal{O})) \\ =& H^1(X, \mathcal{O}(D)). \end{aligned}$$

□

Remark 6. In Lemma 5, it is important that we work with the open subscheme $H(X)$ of the Hilbert scheme that parameterizes one-dimensional subschemes $D \subset X$ that are pure and have no embedded points. Indeed, otherwise its ideal sheaf $\mathcal{O}(-D)$ fails to be reflexive and thus $\mathit{Ext}^1(\mathcal{O}(-D), \mathcal{O}) \neq 0$ by [Har94, Theorem 1.9]. (Loc. cite implies that $\mathit{Ext}^i(\mathcal{O}(-D), \mathcal{O}) \neq 0$ for either $i = 0$ or 1, but we must have $\mathit{Ext}^0(\mathcal{O}(-D), \mathcal{O}) = 0$ because $\mathcal{O}(-D)$ is torsion-free.) The vanishing of $\mathit{Ext}^1(\mathcal{O}(-D), \mathcal{O})$ was needed in the proof of Lemma 5.

Not only does the proof of Lemma 5 fail, but the rule $D \mapsto \mathit{Hom}(\mathcal{O}(-D), \mathcal{O})$ does not define a morphism $\mathrm{Hilb}(X) \to \overline{M}(X)$. Consider the family of subschemes $\mathcal{D} \subset X \times_k \mathrm{Spec}(k[t])$ defined by the homogeneous ideal $(Y - W, X - Z) \cap (Y + tW, X - t^2 Z)$. For $t_0 \in k, t_0 \neq 1$, the fiber of $\mathcal{D}$ over t_0 is the union D_{t_0} of two disjoint lines lying on X. In particular, the ideal $\mathcal{O}(-D_{t_0})$ is a rank 1 reflexive sheaf. However, for $t_0 = 1$, the subscheme is the union of the two lines $\{(Y - W)(Y + W) = Z = 0\}$ together with an embedded point at the origin. The dual $\mathcal{O}(D_1) := \mathit{Hom}(\mathcal{O}(-D_1), \mathcal{O})$ is the sheaf associated to just the two lines, i.e. $\mathcal{O}(H - E)$. Indeed, the union of the exception divisor and the two lines $\{(Y - W)(Y + W) = X - Z = 0\}$ is a hyperplane section, and the natural inclusion $\mathcal{O}_{-D_1} \to \mathcal{O}_{E-H}$ induces an homomorphism $\mathcal{O}(D_1) \to \mathcal{O}(H - E)$. This homomorphism must be an isomorphism since it is an isomorphism away from the point $[0, 1, 0, 0]$ and bother sheaves are reflexive. Now the Euler characteristic of $\mathcal{O}(D_{t_0})$ is 2, but the Euler characteristic of $\mathcal{O}(D_0)$ is 1, so these two sheaves cannot be the fibers of a flat family of sheaves over a connected base. (Use [Har94, Proposition 2.9] to compute the Euler characteristic.) In particular, there is no morphism $\text{Hilb}(X) \to \overline{M}(X)$ that sends $D_t \to \mathcal{H}om(\mathcal{O}(-D), \mathcal{O})$ for all t.

Almost Cartier Divisors. Here we study the locus in $\overline{M}(X)$ that corresponds to the almost Cartier divisors. Our goal is to prove the parts of the main theorem that concern these divisors: (1) the result that the sheaves $\mathcal{O}(D)$ satisfying $\phi_1(\mathcal{O}(D)) = (a, b)$, $h(\mathcal{O}(D)) = n$ for an n-dimensional irreducible component and (2) the result that the sheaves satisfying $\phi(\mathcal{O}(D)) = (1, 1; 1)$ are embedded points of $\overline{M}(X)$.

We begin by computing the tangent space to $\overline{M}(X)$ at a general point. Recall that, by definition, the tangent space to $\overline{M}(X)$ at $\mathcal{O}(D)$ is equal to the set of first order deformations or equivalently the cohomology group $\text{Ext}^1(\mathcal{O}(D), \mathcal{O}(D))$.

Lemma 7. *If* $\mathcal{O}(D) \in \text{APic}(X)$*, then* $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D)) = \mathcal{O}_X$.

Proof. When $\mathcal{O}(D)$ is a line bundle, this is just the identity $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D)) = \mathcal{O}(D)^\vee \otimes \mathcal{O}(D) = \mathcal{O}_X$. In general, observe that $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D))$ is naturally isomorphic to an algebra extension of $\mathcal{O}_X$ contained in the field of rational functions $k(X)$ on X. (Embed $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D))$ by sending an endomorphism f to $f(s)$ for s a fixed generator of the stalk of $\mathcal{O}(D)$ at the generic point.) Furthermore, as an algebra extension of $\mathcal{O}_{\widetilde{X}}$, $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D))$ is integral since, if we can pick a presentation of $\mathcal{O}(D)$, then we can represent a given endomorphism by a matrix and then apply the Hamilton–Cayley theorem.

The only integral extensions of $\mathcal{O}_X$ are $\pi_*\mathcal{O}_{\widetilde{X}}$ and $\mathcal{O}_X$ itself. Away from the finite set of points where $\mathcal{O}(D)$ fails to be locally free, $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D))$ is isomorphic to $\mathcal{O}_X$, so $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D))$ cannot equal $\pi_*\mathcal{O}_{\widetilde{X}}$. We conclude that $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D)) = \mathcal{O}_X$, as desired. ☐

Lemma 8. *If* $\mathcal{O}(D) \in \text{APic}(X)$*, then the natural maps*

$$\text{Ext}^i(\mathcal{O}(D), \mathcal{O}(D)) \to H^0(\mathcal{E}xt^i(\mathcal{O}(D), \mathcal{O}(D))$$

are isomorphisms.

Proof. We use the local-to-global spectral sequence

$$E_2^{p,q} = H^p(\mathcal{E}xt^q(\mathcal{O}(D), \mathcal{O}(D)) \Rightarrow \text{Ext}^{p+q}(\mathcal{O}(D), \mathcal{O}(D)).$$

For $i > 0$, the sheaf $\mathcal{E}xt^i(\mathcal{O}(D), \mathcal{O}(D))$ is supported on the locus where $\mathcal{O}(D)$ failed to be locally free. Since this is a finite set, we have $\mathcal{E}xt^i(\mathcal{O}(D), \mathcal{O}(D)) = 0$ for $i > 0$. For $i = 0$, Lemma 7 states that $\mathcal{H}om(\mathcal{O}(D), \mathcal{O}(D)) = \mathcal{O}$, and $H^i(X, \mathcal{O}) = 0$ for $i > 0$ by Equation (1). We conclude from the spectral sequence that the natural maps $H^0(\mathcal{E}xt^i(\mathcal{O}(D), \mathcal{O}(D)) \to \text{Ext}^1(\mathcal{O}(D), \mathcal{O}(D))$ are isomorphisms. ☐

Proposition 9. *Suppose that* $\mathcal{O}(D) \in \text{APic}(X)$ *satisfies* $\phi_2(\mathcal{O}(D)) = [\alpha_0]$ *for* $\alpha_0 \in \text{Div}(\widetilde{D}_{sing})$ *an effective divisor with support disjoint from the pinch points* $[0, 1, 0]$ *and* $[0, 0, 1]$*. Then*

$$\dim \text{Ext}^1(\mathcal{O}(D), \mathcal{O}(D)) = h([\alpha_0]).$$

Proof. By Lemma 8, it is equivalent to show that $h^0(\mathcal{E}xt^1(\mathcal{O}(D), \mathcal{O}(D)) = h([\alpha_0]))$. The group $H^0(\mathcal{E}xt^1(\mathcal{O}(D), \mathcal{O}(D)))$ breaks up as a direct sum over its zero dimensional support. The stalk of $\mathcal{E}xt^1(\mathcal{O}(D), \mathcal{O}(D))$ at p remains unchanged if we pass from X to the completed local ring $\widehat{\mathcal{O}}_{X,p}$. Since we assumed that the support of α_0 does not contain a pinch point, $\widehat{\mathcal{O}}_{X,p_0}$ is isomorphic to $k[x, y, z]/xy$. By the classification of rank 1 reflexive sheaves, the completed stalk of $\mathcal{O}(D)$ must be isomorphic to the ideal $M := (y, z^n)$ for some n. We will prove the theorem by showing that $n = \dim \operatorname{Ext}^1(M, M)$.

From the theory of matrix factorizations, a free resolution of M is

$$\cdots \to \mathcal{O}^2 \xrightarrow{B} \mathcal{O}^2 \xrightarrow{A} \mathcal{O}^2 \to M \to 0,$$

where

$$A := \begin{pmatrix} x & z^n \\ 0 & -y \end{pmatrix} \text{ and } B := \begin{pmatrix} y & z^n \\ 0 & -x \end{pmatrix}.$$

Thus $\operatorname{Ext}^1(M, M)$ is computed as the first homology group of the complex

$$\cdots \leftarrow M^2 \xleftarrow{\operatorname{Tr}(B)} M^2 \xleftarrow{\operatorname{Tr}(A)} M^2 \leftarrow 0.$$

Here $\operatorname{Tr}(A)$ denotes the transpose.

An elementary computation shows that the homology group $\ker(\operatorname{Tr}(B))/\operatorname{coker}(\operatorname{Tr}(A))$ has as k-basis the elements $(0, x), (0, zx), \ldots, (0, z^{n-1}x)$. In particular, the dimension is n. □

Lemma 10. *The Abel map* $\operatorname{Abel}\colon H(X) \to \overline{M}(X)$ *induces an isomorphism over the locus of sheaves* $\mathcal{O}(D) \in \operatorname{APic}(X)$ *satisfying* $\phi_1(X) = (1, 1)$ *and* $h(\mathcal{O}(D)) = 1$. *This common scheme is a nonreduced curve isomorphic to the scheme obtained from gluing*

$$U_1 := \operatorname{Spec}(k[a_1, b_1, c_1, d_1]/(a_1, b_1 - c_1^2, c_1 d_1, d_1^2)) \tag{2}$$

to

$$U_2 := \operatorname{Spec}(k[a_2, b_2, c_2, d_2]/(c_2, a_2^2 - d_2, a_2 b_2, b_2^2)) \tag{3}$$

along $U_1 - \{0\} \to U_2 - \{0\}$ *via the morphism induced by* $a_2 \mapsto -1/c_1$.

The two sheaves satisfying $\phi_2(\mathcal{O}(D)) =$ *[pinch point] correspond to the origin in* U_1 *and the origin in* U_2.

Proof. Suppose that $\mathcal{O}(D) \in \operatorname{APic}(X)$ satisfies $\phi_1(X) = (1, 1)$ and $h(\mathcal{O}(D)) = 1$. Then

$$h^i(X, \mathcal{O}(D)) = \begin{cases} 1 & \text{if } i = 0; \\ 0 & \text{if } i > 0. \end{cases} \tag{4}$$

To see this, observe that D can be taken to a suitable ruling of X. Since this divisor is effective, $h^0(X, \mathcal{O}(D)) \geq 1$, and $h^2(X, \mathcal{O}(D)) = 0$ by coherent duality. (We have $h^2(X, \mathcal{O}(D)) = h^0(X, \omega_X \otimes \mathcal{O}(-D))$. This second group must vanish since $\omega_X \otimes \mathcal{O}(-D) = \mathcal{O}(-H - D)$ is the inverse of a non-empty effective divisor.) From the following short exact sequence (taken from [Har94, Proposition 2.9])

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to \omega_D \otimes \omega_X^{-1} \to 0,$$

we get that $\chi(X, \mathcal{O}(D)) = 1$, so we must have $h^1(X, \mathcal{O}(D)) = 0$.

From Lemma 5, we deduce that the Abel map $\mathrm{Hilb}(X) \to \overline{M}(X)$ is an isomorphism over a Zariski open neighborhood containing the subset of interest. We complete the proof by explicitly computing on $\mathrm{Hilb}(X)$. Subschemes $D \subset X$ with $\phi(\mathcal{O}(D)) = (1, 1)$ and $h(\mathcal{O}(D)) = 1$ are lines contained in X. Consider the equations

$$X - a_1 W - b_1 Z = Y - c_1 W - d_1 Z = 0.$$

They define an open embedding $\mathrm{Spec}(k[a_1, b_1, c_1, d_1]) \to \mathrm{Gr}(3, 1)$ in the Grassmannian of lines, and we compute the intersection $\mathrm{Spec}(k[a_1, b_1, c_1, d_1]) \cap \mathrm{Hilb}(X)$. It is defined by the equations that are the coefficients of the following equations:

$$(S^2(a_1 S + b_1 T) - (c_1 S + d_1 T)^2 S)$$

The description for U_1 is given by expanding out this expression. The scheme U_2 is obtained by working instead with the equations

$$W - a_2 Y - b_2 X = Z - c_2 Y - d_2 X = 0.$$

□

Corollary 11. *The moduli space $\overline{M}(X)$ has an embedded point at the sheaves $\mathcal{O}(D)$ satisfying $\phi(\mathcal{O}(D)) = (1, 1; [\textit{pinch point}])$.*

Proof. This follows immediate from Equations (2) and (3). □

Lemma 12. *Suppose that $L \subset X$ is the ruling defined by the homogeneous ideal generated by the polynomials $\alpha W - Y, X - \alpha^2 Z$ with $\alpha \in k$, $\alpha \neq 0$.*

Let $n > 0$ be a given integer. Then the multiple $n \cdot L$ is the effective divisor defined by the ideal generated by the polynomials

$$NF^{n-2}, NF^{n-3}G, \ldots, NF^{n-i-i}G^i, \ldots, NG^{n-2}, G^n. \tag{5}$$

Here

$$F := \alpha W - Y, G := X - \alpha^2 Z, N := WG + 2\alpha F Z.$$

Proof. Recall that, by definition, the ideal of $n \cdot L$ is the reflexive hull of the n-th power of the ideal of L. Temporarily set J_n equal to the ideal of the subscheme defined by the homogeneous ideal (5) and $I = I_L$ equal to the ideal of L. With this notation, our goal is to prove that J_n is the reflexive hull of I_L^n.

Observe that the polynomial N is defined so that

$$\begin{aligned} WN =& W^2 G + 2\alpha W F \\ =& W^2 X - \alpha^2 W^2 + (\alpha W + Y + F)F \\ =& W^2 X - \alpha^2 W^2 + (\alpha W + Y)(\alpha W - Y) + F^2 \\ =& W^2 X - \alpha^2 W^2 + \alpha^2 W^2 - Y^2 + F^2 \\ =& (W^2 X - Y^2 Z) + F^2 \\ =& F^2 \text{ on the surface } X. \end{aligned}$$

Now observe that we have the containment $I_L^n \subset J_n$ because

$$\begin{aligned} F^n =& F^2F^{n-2}, \\ =& WNF^{n-2}, \\ F^{n-1}G =& WNF^{n-3}G, \\ & \dots\dots \\ F^{n-i}G^i =& WNF^{n-i-2}G^i, \\ & \dots\dots \\ F^2G^{n-2} =& WNG^{n-2}, \\ FG^{n-1} =& 1/(2\alpha)NG^{n-1} - 1/(2\alpha)WG^n. \end{aligned}$$

These equations also show that the containment $I_L^n \subset J_n$ becomes an equality after restricting to the complement of $\{W = 0\}$. Trivially, the containment is an equality away from the support of $\mathcal{O}_X/J_n$, so the two ideals are equal away from the union of that support and $\{W \neq 0\}$, and this is just the closed point given in projective coordinates by $[0, \alpha^2, 0, 1]$. Since this subset has codimension 2, we deduce that I_L^n and J_n have the same reflexive hull.

To complete the proof, we need to show that J_n is reflexive. In fact, it is enough to show this holds for the restriction of J_n to the affine open $\mathrm{Spec}(k[w, x, y]/w^2x - y^2)$ (i.e. the complement of $\{Z = 0\}$) since I_L^n is automatically reflexive away from the singular locus. By abuse of notation, let J_n also denote the ideal in $k[w, x, y]/w^2x - y^2$. To show that J_n is reflexive, it is enough to construct an isomorphism

$$k[w, x, y]/(J_n + w^2x - y^2) \cong k[s, t]/(t - \alpha)^n. \tag{6}$$

Indeed, given such an isomorphism, we deduce that the ideal J_n defines a subscheme of pure codimenion 1 and hence is reflexive.

To show (6), consider the the homomorphism

$$\begin{gathered} \phi \colon \mathcal{O} \to k[s, t], \\ w \mapsto s, x \mapsto t^2, y \mapsto st. \end{gathered}$$

An algebra computation show that ϕ induces a homomorphism $\overline{\phi} \colon \mathcal{O}/J_n \to k[s, t]/(t - \alpha)^n$. To see that $\overline{\phi}$ is an isomorphism, observe that $\mathcal{O}/J_m$ contains a square root of x since it can be expressed as the appropriate truncated power series:

$$\begin{aligned} x^{1/2} &= \left(\alpha^2 + x - \alpha^2\right)^{1/2} \\ &= \sum_{i=0}^{n-1} \binom{1/2}{i} \alpha^{1-i}(1 - \alpha^2)^i \\ &= \sum_{i=0}^{n-1} \binom{1/2}{i} \alpha^{1-i} G^i. \end{aligned}$$

Thus we have a well-defined homomorphism $\overline{\psi}\colon k[s,t]/(t-\alpha)^n \to \mathcal{O}/J_n$ defined by

$$\begin{aligned}\overline{\psi}(s) =& w,\\ \overline{\psi}(t) =& \sum_{i=0}^{n-1} \binom{1/2}{i} \alpha^{1-i} G^i.\end{aligned}$$

The homomorphism $\overline{\psi}$ is the inverse of $\overline{\phi}$ because

$$\begin{aligned}\overline{\psi}(\overline{\phi}(w)) =& \overline{\psi}(s)\\ =& w\\ \overline{\psi}(\overline{\phi}(x)) =& \overline{\psi}(t^2)\\ =& \left(\sum_{i=0}^{n-1} \binom{1/2}{i} \alpha^{1-i} G^i\right)^2\\ \overline{\psi}(\overline{\phi}(y)) =& \overline{\psi}(st)\\ =& w \cdot \left(\sum_{i=0}^{n-1} \binom{1/2}{i} \alpha^{1-i} G^i\right)\end{aligned}$$

□

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Lemma 13. *Let $n > 0$ be an integer and $L \subset X$ be a ruling that does not pass through a pinch point. Then there exists a $k[t]$-flat subscheme $\mathcal{D} \subset X \times_k \mathbf{A}^1_k$ such that the fiber over $0 \in \mathbf{A}^1_k$ is $n \cdot L$ and the fiber over a general point is n disticnt rulings of X.*

Proof. We will construct a family over $\mathrm{Spec} k[t]$ such that the fiber over 0 is $n \cdot L$ and the general fiber is the union of n distinct lines.

By Lemma 12, the subscheme $n \cdot L$ is defined by the homogeneous ideal generated by the polynomials

$$NF^{n-2}, NF^{n-3}G, \ldots, NF^{n-i-i}G^i, \ldots, NG^{n-2}, \text{ and } G^n.$$

Fix n distinct nonzero scalars $\beta_1, \beta_2, \ldots, \beta_n \in k$. Then set

$$\begin{aligned}\ell_1(a) =& \beta_1 t + \alpha,\\ \ell_2(a) =& \beta_2 t + \alpha,\\ & \ldots\\ \ell_n(a) =& \beta_n t + \alpha\end{aligned}$$

and define $\mathcal{D}_i \subset X \times_k \mathbf{A}^1_k$ to be the closed subscheme defined by the homogeneous ideal generated by

$$\ell_i(t)W - Y \text{ and } X - \ell(t)^2 Z.$$

We will show that the union $\mathcal{D}_1 + \ldots \mathcal{D}_n$ satisfies the desired properties.

First, let us show that the fiber of $\mathcal{D}_1 + \cdots + \mathcal{D}_n$ over 0 is contained in $n \cdot L$. We will show containment of the corresponding ideals in the affine neighborhood $\mathrm{Spec}(k[w,x,y]/w^2x - y^2)$, i.e. the neighborhood where $Z \neq 0$. Verifying equality in the other affine neighborhoods is easier since there all the subschemes are Cartier, and we leave the details to the interested reader.

On $\mathrm{Spec}(k[w,x,y]/w^2x - y^2)$, the ideal of $n \cdot L$ is generated by the polynomials

$$N(w,x,y,1)F^{n-2}(w,x,y,1), N(w,x,y,2)F^{n-2}(w,x,y,1)G(w,x,y,1), \ldots,$$
$$N(w,x,y,1)G^{n-2}(w,x,y,z), \text{ and } G^n(w,x,y,1).$$

To ease notation, we denote the polynomials $N(w,x,y,1)$, $F(w,x,y,1)$, $G(w,x,y,1)$ by N, F, G.

Consider first the case where $n = 2$. The subscheme $\mathcal{D}_1 + \mathcal{D}_2$ is defined by the ideal

$$\left(\ell_1(t)w - y, x - \ell_1(t)^2\right) \cap \left(\ell_2(t)w - y, x - \ell_2(t)^2\right).$$

This intersection contains the product $\left(x - \ell_1(t)^2)\right) \cdot \left(x - \ell_2(t)^2)\right)$, so the ideal of the special fiber of $\mathcal{D}_1 + \mathcal{D}_2$ contains the specialization $\left(x - \ell_1(0)^2)\right) \cdot \left(x - \ell_2(0)^2)\right) = G^2$.

Another element in the intersection is the polynomial $(\ell_1(t) \cdot \ell_2(t)w - (\ell_1(t) + \ell_2(t))y + wx$. Indeed, the expression

$$\ell_1(t) \cdot \ell_2(t)w - (\ell_1(t) + \ell_2(t))y + wx = (\ell_1(t) + \ell_2(a))(\ell_1(t)w - y) + w(x - \ell_1^2(t)).$$

shows that the element lies in the first term in the intersection. Reversing the roles of ℓ_1 and ℓ_2, we see that the ideal also lies in the second term. We conclude that the ideal of the fiber of $\mathcal{D}_1 + \mathcal{D}_2$ contains

$$\begin{aligned}(\ell_1(0) \cdot \ell_2(0)w - (\ell_1(0) + \ell_0(t))y + wx =& \alpha^2 w - 2\alpha y + wx \\ =& N.\end{aligned}$$

This shows that the ideal of the fiber of $\mathcal{D}_1 + \mathcal{D}_2$ contains the ideal of $2 \cdot L$ or equivalently $(\mathcal{D}_1 + \mathcal{D}_2) \cap X \times_k \{0\} \subset 2 \cdot L$. The subscheme $2 \cdot L$ has the same Hilbert polynomial as the general fiber of $\mathcal{D}_1 + \mathcal{D}_2$. We not yet shown that $\mathcal{D}_1 + \mathcal{D}_2$ is $k[t]$-flat, but we can argue as follows. The restriction of $\mathcal{D}_1 + \mathcal{D}_2 \to \mathbf{A}^1_k$ to $\mathbf{A}^1_k - \{0\}$ is $k[t, t^{-1}]$-flat because it is just a family of n disjoint lines. Thus if we let $\overline{D}$ equal to the Zariski closure of the generic fiber in $\mathcal{D}_1 + \mathcal{D}_2$, then $\overline{\mathcal{D}}$ is $k[t]$-flat and contained in $\mathcal{D}_1 + \mathcal{D}_2$. By flatness, the fiber of $\overline{\mathcal{D}}$ over 0 has the same Hilbert polynomial as the disjoint union of n lines. This is the same as the Hilbert polynomial of $2 \cdot L$, so the inclusion $\overline{\mathcal{D}} \cap X \times_k \{0\} \subset 2 \cdot L$ must be an equality. We deduce that $\overline{\mathcal{D}} = \mathcal{D}_1 + \mathcal{D}_2$, so $\mathcal{D}_1 + \mathcal{D}_2$ has the desired properties.

Now suppose that n is arbitrary. Then $\mathcal{D}_1 + \cdots + \mathcal{D}_n$ is defined by

$$\left(\ell_1(t)w - y, x - \ell_1(t)^2\right) \cap \cdots \cap \left(\ell_n(t)w - y, x - \ell_n(t)^2\right).$$

We have just shown that $\ell_{n-1}(t) \cdot \ell_n(t)w - (\ell_{n-1}(t) + \ell_n(t))y + wx$ lies in the intersection of the last two ideals. Thus if we take the product of this element with $n-2$ elements, with the i-th being either or $\ell_i(t)w - y$ or $x - \ell_i(t)^2$, then we obtain an element of the ideal of $\mathcal{D}_1 + \cdots + \mathcal{D}_n$. Passing to the special fiber, we get every element of the form NF^iG^{n-2-i}. We get the element G^n by taking the product of $\ell_i(t)w - y$ for $i = 1, 2, \ldots, n$. We complete the proof by arguing as in the $n = 2$ case. $\square$

Corollary 14. *Suppose that we are given rulings $L_1, \dots, L_k$ of X that do not pass through pinch points and integers $n_1, \dots, n_k > 0$.*

Then there exists a nonempty open neighborhood $U \subset k[t]$ of $0 \in \mathbf{A}^1_k$ and a $k[t]$-flat subscheme $\mathcal{D} \subset X \times_k \mathbf{A}^1_k$ such that the fiber over $0 \in \mathbf{A}^1_k$ is $n \cdot L$ and the fiber over a general point is n disticnt rulings of X.

Proof. By grouping n_i's, we can assume that the rulings $L_1, \dots, L_k$ are distinct. Let $\mathcal{D}_i \subset X \times_k \mathbf{A}^1_k$ be a $k[t]$ flat-family of subschemes such that the fiber over a general point if n_i distinct lines and the fiber over 0 is $n_i \cdot L_i$, i.e. the family constructed in Lemma 13. Define $\mathcal{D} \subset X \times_k \mathbf{A}^1_k$ to be the union of $\mathcal{D}_1, \dots, \mathcal{D}_k$ (i.e. the subscheme defined by the intersection of the corresponding ideals).

There could be points where two families $\mathcal{D}_i$ and $\mathcal{D}_j$ intersect, and at such a point, it isn't entirely clear that $\mathcal{D} \to \mathbf{A}^1_k$ is $k[t]$-flat. To address this, observe that $\mathcal{D}_1 \cap \dots \cap \mathcal{D}_k$ is a closed subset of $X \times_k \mathbf{A}^1_k$ that does any points lying above the origin. The projection $\mathrm{pr}_2 \colon X \times_k \mathbf{A}^1_k \to \mathbf{A}^1_k$ is proper, so $\mathrm{pr}_2(\mathcal{D}_1 \cap \dots \cap \mathcal{D}_k)$ is a closed subset that does not contain the origin. Define $U := \mathbf{A}^1_k - \mathrm{pr}_2(\mathcal{D}_1 \cap \dots \cap \mathcal{D}_k)$. □

Lemma 15. *Within the subset $\mathrm{APic}(X)$ of $\overline{M}(X)$, the locus of sheaves with the property that $\phi_2(\mathcal{O}(D))$ can be represented by a reduced effective divisor $\alpha \in \mathrm{Div}(\widetilde{D}_{sing})$ are dense.*

Proof. Lemma 14 shows that all sheaves of the form $\mathcal{O}(n_1 \cdot L_1 + \dots + n_k L_k)$ where $n_1, \dots, n_k$ are positive integers and $L_1, \dots L_k$ are ruling that do not pass through pinch points.

Now suppose that $\mathcal{O}(D)$ is arbitrary. Write $\phi(\mathcal{O}(D)) = (n, m, [\alpha_0])$ for $n = \deg(\alpha_0)$ and α_0 an effective divisor of minimal degree. Write $\alpha_0 = n_1 p_1 + \dots + n_k p_k + q_1 + \dots + q_l$, where $n_i > 1$ and the points $p_1, \dots, p_k, q_1, \dots, q_l$ are distinct. From the classification of rank 1 reflexive sheaves on the fold singularity, we deduce that the pinch points are not among the points $p_1, \dots, p_k$.

For $i = 1, \dots, k$, pick a ruling L_i such that $\phi(\mathcal{O}(L_i)) = (1, 1, [p_i])$. Then write $n = \deg(\alpha_0) + 2a$ and set $b = \deg(\alpha_0) - m + a$. Consider the sheaf $\mathcal{O}(D - aH - bE)$ where H is a hyperplane section and E is the image of the exceptional divisor of $\widetilde{X}$. This sheaf satisfies $\phi(\mathcal{O}(D - aH - bE)) = (\deg(\alpha_0), \deg(\alpha_0), [\alpha_0])$. Thus $\mathcal{O}(D - aH - bE)$ has the same image under ϕ as $\mathcal{O}(n_1 L_1 + \dots + n_k L_k + M_1 + \dots + M_l)$. We have already shown that this last sheaf is a fiber of a family over $U \subset \mathbf{A}^1_k$ with the property that the general fiber is a disjoint union of rulings. By tensoring this family with the constant family of line bundles with fiber $\mathcal{O}(aH + bE)$, we obtain a family where the one fiber is $\mathcal{O}(D)$ and the other fibers have that the image under ϕ_2 can be represented by a reduced effective divisor. In particular, $\mathcal{O}(D)$ is in the closure of the locus of sheaves $\mathcal{O}(D')$ such that $\phi_2(\mathcal{O}(D'))$ can be represented by a reduced effective divisor. □

Proposition 16. *Two elements $\mathcal{O}(D_1), \mathcal{O}(D_2) \in \mathrm{APic}(X)$ of the almost Picard group lie in the same irreducible component if and only if $\phi_1(\mathcal{O}(D_1)) = \phi_1(\mathcal{O}(D_2))$ and $h(\mathcal{O}(D_1)) = h(\mathcal{O}(D_2))$. This common component has dimension $h(\mathcal{O}(D))$ and is regular at every $\mathcal{O}(D)$ that has the property that $\phi_1(\mathcal{O}(D))$ can be represented by an effective divisor with support disjoint from the pinch points.*

Proof. Let $\mathcal{O}(D) \in \mathrm{APic}(X)$ be given. Set $(a, b) = \phi_1(\mathcal{O}(D))$ and $\alpha_0 := \alpha(\mathcal{O}(D))$. Write $c = (n - \deg(\alpha_0))/2$ and $d = \deg(\alpha_0) + a - b$.

Set $U \subset H(X)$ equal to the reduced subscheme Hilbert scheme of rulings on X. This is the reduced subscheme of the scheme appearing in Lemma 10 (so it is a rational curve $\cong \mathbf{P}^1_k$). Consider the locus $V \subset U \times_k \cdots \times_k U$ of distinct rulings inside the α_0-fold self-product of U. The rule $(L_1, \dots L_k) \mapsto \mathcal{O}(L_1 + \cdots + L_K + cH + dE)$ defines a morphism $V \to \overline{M}(V)$. By classification, the image contains all sheaves $\mathcal{O}(D)$ such that $\phi_1(\mathcal{O}(D)) = (a, b)$ and $\phi_2(\mathcal{O}(D))$ is a reduced divisor of degree α_0. The Zariski closure contains all sheaves where $\phi_2(\mathcal{O}(D))$ is possibly nonreduced by Lemma 15. We conclude that $\mathcal{O}(D)$ lies on an irreducible component of $\overline{M}(X)$ of dimension $\geq \deg(\alpha_0)$.

We get the reverse inequality, dimension $\leq \deg(\alpha_0)$, using Proposition 9. That proposition computes the tangent space to $\overline{M}(X)$ at a point where $\phi_1(\mathcal{O}(D)) = (a, b)$ and $\phi_2(\mathcal{O}(D))$ can be represented by a reduced effective divisor of degree $\deg(\alpha_0)$ as $\deg(\alpha_0)$. We conclude that the dimension of the irreducible component of $\deg(\alpha_0)$.

Now suppose further that the support of α_0 is disjoint from the pinch points. Then Proposition 9 states that the tangent space dimension to $\overline{M}(X)$ at $\mathcal{O}(D)$ is $h(\mathcal{O}(D))$. Since this equals the local dimension of $\overline{M}(X)$, we conclude that the moduli space is regular at $\mathcal{O}(D)$. □

Remark 17. When $(a, b) = (1, 1)$ and $n = 1$, Lemma 10 implies the stronger result that the rank 1 reflexive sheaves satisfying $\phi_1(\mathcal{O}(D)) = (1, 1)$ and $h(\mathcal{O}(D)) = 1$ form a connected component, not just an irreducible component. This is not always the case. Consider the case where $(a, b) = (-2, -2)$ and $n = 2$. Then the irreducible component in question meets a component containing the ideal sheaves of the union of a hyperplane and a point.

To see this, consider the ideal of the closed subscheme defined by the homogeneous ideal I spanned by $(Y - W)(Y + W)$, $(Y - W)(X - Z)$, $(X - Z)(Y + W)$, and $(X - Z)^2$. This subscheme is the hyperplane section $\{X - Z = 0\}$ together with an embedded point at $[0, 0, 1, 1]$. The flat family over $\mathrm{Spec}(k[t])$ defined by the homogeneous ideal $(X - Z) \cap (X, Y, Z - t)$ realizes I as the limit of the ideal of a union of a hyperplane and a point. The homogeneous ideal $(Y - W, X - Z) \cap (Y + tW, X - t^2 Z)$ realizes it as the limit of a rank 1 reflexive sheaf $\mathcal{O}(D)$ satisfying the $\phi_1(\mathcal{O}(D)) = (-2, -2)$ and $h(\mathcal{O}(D)) = 2$.

Observe that the ideal of the union of the hyperplane and the embedded point is not reflexive (since the subscheme it defines is not Cohen–Macaulay). Thus this construction shows that the locus of reflexive sheaves is not closed in $\overline{M}(X)$.

This example also shows that a connected component of $\overline{M}(X)$ can fail to be equidimensional. Indeed, the irreducible component containing the reflexive sheaves satisfying $\phi_1(\mathcal{O}(D)) = (-2, -2)$ and $h(\mathcal{O}(D)) = 2$ meets the component containing the ideal sheaves of subschemes consisting of a hyperplane section and a disjoint point. The first component has dimension two, while the dimension of the second component is bounded below by $3 + 2 = 5$.

Example 18. To illustrate what we have proven, consider the twisted cubic curve D on X that is the image of $\mathbf{P}^1_k \to X$ under the morphism defined in projective coordinates by

$[S, T] \mapsto [T(T+S)^2, T^2S, T^2(S+T), (T+S)^2S]$. This is one of the examples of a smooth set-theoretic complete intersection contained in X that is given in [HP15, Example 7.12]. All such curves are given by the image $D_{a,b,c}$ of the morphism $\mathbf{P}^1_k \to \mathbf{P}^3_k$ defined by

$$[S, T] \mapsto [T(cT+bS)^2, a^2T^2S, -aT^2(cT+bS), (cT+bS)^2S].$$

Here $a, b, c \in k$ are nonzero scalars. All these curves are linearly equivalent to D. Indeed, $\phi(D_{a,b,c}) = (2, 1; [\text{ pinch points}]$. A Riemann–Roch computations shows that the complete linear system of effective almost Cartier divisors equivalent to D has dimension ≥ 2. We conclude that the dimension is exactly 2, and the curves $D_{a,b,c}$ fill out a Zariski dense subset.

There are also singular curves linearly equivalent to D. Let L_1 equal the ruling that passes through the pinch point $[0, 1, 0, 0]$ and L_2 the ruling passing through $[0, 0, 0, 1]$. Let C_1 denote the nonreduced curve defined by the projective ideal $(W, Y)^2 + (Y)$ and C_2 the curve defined by $(W, Y)^2 + (W)$. Then the curves $L_1 + C_2$, $L_2 + C_1$, and $L_1 + L_2 + E$ are all linearly equivalent to D.

The complete linear system associated to D is not a component of the Hilbert scheme $\text{Hilb}(X)$. Indeed, Proposition 16 states that $\mathcal{O}(D)$ lies on a 2-dimensional component of $\overline{M}(X)$. Over a Zariski open neighborhood of $\mathcal{O}(D)$, the Abel map $\text{Hilb}(X) \to \overline{M}(X)$ has 2-dimensional fibers, so D lies in a 4-dimensional component of $\text{Hilb}(X)$. This component can be described explicitly. For general scalars $a, b, c, d \in k$, the image of the morphism $\mathbf{P}^1_k \to \mathbf{P}^3_k$ defined by

$$[S, T] \mapsto [-(bT^2 + cTS + dS^2)S, (T+aS)T^2, -T(bT^2 + cTS + dS^2), (T+aS)S^2]$$

is a curve on X that lies in the same component of $\text{Hilb}(X)$ as D. The subset of all such curves is dense in the irreducible component containing D.

The singular locus. To complete our analysis of the reflexive sheaves in $\overline{M}(X)$, we need to describe the irreducible components containing elements of $\text{GPic}(X) - \text{APic}(X)$. If $\mathcal{O}(D) \in \text{GPic}(X) - \text{APic}(X)$, then we will show that $\{\mathcal{O}(D)\}$ is the support of a nonreduced component of $\overline{M}(X)$. We prove this by first proving the analogous statement for the Hilbert scheme and then completing the proof using the Abel map.

Recall that the singular locus $D_{\text{sing}} \subset X$ defines an element $\mathcal{O}(D_{\text{sing}})$ of $\text{GPic}(X) - \text{APic}(X)$, and every element of $\text{GPic}(X) - \text{APic}(X)$ is of the form $\mathcal{O}(C + D_{\text{sing}})$ for some almost Cartier divisor C. We first study $\mathcal{O}(D_{\text{sing}})$.

Lemma 19. *The curve D_{sing} is the support of a connected component of* $\text{Hilb}(X)$ *that is isomorphic to*

$$\text{Spec}(k[a, b, c, d]/(a^2, 2ab - c^2, b^2 - cd, d^2)).$$

In particular, this component is nonreduced and supported at a point.

Proof. By local constancy of the degree and arithmetic genus, the subscheme of lines in X is a union of connected components of $\text{Hilb}(X)$ that contains D_{sing}. This subscheme is naturally a closed subscheme of the Grassmannian $\text{Gr}(1, 3)$ of lines in $\mathbf{P}^3_k$, and we prove the lemma by explicitly computing equations for it.

The equations

$$W - aX - bZ = Y - CX - DZ = 0. \tag{7}$$

define a family of lines parameterized by $\operatorname{Spec}(k[a,b,c,d])$ and hence a morphism

$$\operatorname{Spec}(k[a,b,c,d]) \to \operatorname{Gr}(1,3).$$

Furthermore, the origin $0 \in \operatorname{Spec}(k[a,b,c,d])$ corresponds to the line D_{sing}, so the family of lines defines an open immersion $\operatorname{Spec}(k[a,b,c,d]) \to \operatorname{Gr}(3,1)$ with image a Zariski open neighborhood of D_{sing} in $\operatorname{Gr}(1,3)$. We now compute the intersection of this neighborhood with $\operatorname{Hilb}(X)$.

A family of parameterizations of the family of lines in (7) is given in projective coordinates by $[S,T] \mapsto [aS+bT, S, cS+dT, T]$. The intersection $\operatorname{Spec}(k[a,b,c,d]) \cap \operatorname{Hilb}(X)$ is defined by the equations that are the coefficients of the equation obtained by substituting the parameterization into the equation for X:

$$(aS+bT)^2 S - (cS+dT)^2.$$

The lemma follows by expanding out this polynomial. □

Lemma 20. *As* L *varies over the rulings of* X*, the curves* $L_p + D_{sing}$ *form the closed points of a connected component of the Hilbert scheme* $\operatorname{Hilb}(X)$.

This component is covered by two Zariski open affine subscheme U_1 *and* U_2*, each of which is isomorphic to* $\operatorname{Spec}(k[a,b,c,d,e,f,g,h]/I)$ *for* I *the ideal generated by*

$$d, e^2, a^2+f, beg+2abe+eg-b^2, 2abf+gf-ef^2-2ac, \tag{8}$$

$$eh+2abg+g^2-efg-2bc, 2abh+gh-efh-c^2. \tag{9}$$

The isomorphism can be chosen so that $U_1 \cap U_2 \subset U_i$ *corresponds to the complement of the origin in* $\operatorname{Spec}(k[a,b,c,d,e,f,g,h]/I)$.

The origin in U_1 *corresponds to* $L_{p_1} + D_{sing}$ *for* $p_1 = [1,0,0,0]$*; the origin in* U_2 *corresponds to* $L_{p_2} + D_{sing}$ *for* $p_2 = [0,0,1,0]$*. (The rulings* L_{p_1} *and* L_{p_2} *are the ruling that pass through a pinch point.)*

Proof. The proof is similar to the proof of Lemma 19 except that the algebra computation is more involved. The curve $L_x + D_{\text{sing}}$ is a space curve of degree 2 and genus 0. The equations

$$Y - aW - bX - cZ = 0, \tag{10}$$

$$WX + dW^2 + eX^2 + fWZ + gXZ + hZ^2 = 0 \tag{11}$$

define a flat family of such curves and hence a morphism $\operatorname{Spec}(k[a,b,c,d,e,f,g,h]) \to \operatorname{Hilb}(\mathbf{P}^3)$.

Observe that the morphism $\operatorname{Spec}(k[a,b,c,d,e,f,g,h]) \to \operatorname{Hilb}(\mathbf{P}^3)$ is injective on geometric points. Indeed, every curve of degree 2 and genus 0 is a complete intersection defined by a liner polynomial and a quadric polynomial. The linear polynomial is unique up to scaling; the quadratic up to scaling and adding a multiple of the linear equation. We conclude that $\operatorname{Spec}(k[a,b,c,d,e,f,g,h]) \to \operatorname{Hilb}(\mathbf{P}^3)$ is an open immersion. (Use e.g. Zariski's main theorem.)

Now consider the intersection $\mathrm{Spec}(k[a,b,c,d,e,f,g,h]) \cap \mathrm{Hilb}(X)$. The plane (10) is parameterized by

$$[S,T,U] \mapsto [S,T,aS+bT+cU,U].$$

The condition that a subscheme is contained in X is the condition that the following equation, involving two new variables i and j, has a solution:

$$S^2T-(aS+bT+cU)^2 = (ST+dS^2+eT^2+fSU+gSU+hU^2)(S+iT+jU).$$

The equations (8) are obtained from the by expression by expanding out, collecting coefficients, and then eliminating the variables i, j.

The reduced subscheme of $\mathrm{Spec}(k[a,b,c,d,e,f,g,h]/I)$ is isomorphic to

$$\mathrm{Spec}(k[a,b,c,d,e,f,g,h]/(b,c,d,e,f,g,a^2+f).$$

The closed point $(b,c,d,e,f,g,a-\alpha,f+\alpha^2)$ corresponds to the curve L_p+D_{sing} for $p=[1,0,\alpha,0]$. In particular, the Zariski open neighborhood

$$\mathrm{Spec}(k[a,b,c,d,e,f,g,h]/(b,c,d,e,f,g,a^2+f) \subset \mathrm{Hilb}(X)$$

contains all subschemes of the form L_p+D_{sing} except for the case $p=[0,0,1,0]$. We obtain a second neighborhood of this missing curve by swapping the roles of $[1,0,0,0]$ and $[0,0,1,0]$. □

Lemma 21. *The sheaves $\mathcal{O}(D_{sing})$ and $\mathcal{O}(L_x+D_{sing})$ satisfy*

$$h^i(X,\mathcal{O}(D_{sing})) = \begin{cases} 1 & \text{if } i=0; \\ 0 & \text{otherwise} \end{cases}$$

and

$$h^i(X,\mathcal{O}(L_x+D_{sing})) = \begin{cases} 2 & \text{if } i=0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. A local computation shows that $D_{\text{sing}} \subset X$ is the curve defined by the conductor ideal, so $\mathcal{O}(D_{\text{sing}}) = \pi_*\mathcal{O}_{\widetilde{X}}$. Since π is finite, $H^i(X,\mathcal{O}(D_{\text{sing}})) = H^i(\widetilde{X},\mathcal{O})$, and the result follows from [Har77, Lemma 2.4].

We now turn to the sheaf $\mathcal{O}(L_x+D_{\text{sing}})$. Let $\widetilde{L} \subset \widetilde{X}$ be the strict transform of a line under the blow-down map $\widetilde{X} \to \mathbf{P}^2_k$. The argument just given shows that $\mathcal{O}(L_x+D_{\text{sing}})$ and $\pi_*\mathcal{O}(\widetilde{L})$ are isomorphic away from $L_x \cap D_{\text{sing}}$. We conclude from Hartog's extension principle that they are in fact isomorphic. We complete the proof as before. □

Corollary 22. *If $\mathcal{O}(D) \in \mathrm{GPic}(X)-\mathrm{APic}(X)$, then $\{\mathcal{O}(D)\}$ is the support of a nonreduced irreducible component.*

Proof. When $D=D_{\text{sing}}$, Lemma 21 implies that the Abel map $H(X) \to \overline{M}(X)$ is an isomorphism on a Zariski neighbood of $\mathcal{O}(D_{\text{sing}})$, so the claim follows from Lemma 19.

A similar argument holds when $D=L_x+D_{\text{sing}}$: Lemma 21 implies that, over a Zariski open neighborhood of $\mathcal{O}(L_x+D_{\text{sing}})$, $H(X) \to \overline{M}(X)$ is a $\mathbf{P}^1_k$-bundle, and we deduce the claim by Lemma 20.

Every sheaf is either of the form $\mathcal{O}(D_{sing}) \otimes \mathcal{L}$ or $\mathcal{O}(L_x + D_{sing}) \otimes \mathcal{L}$ for some line bundle since $\pi_*(\mathcal{O}(D)) \otimes \mathcal{L} = \pi_*(\mathcal{O}(D) \otimes \pi^*\mathcal{L})$. Since tensoring with a line bundle defines an automorphism $\overline{M}(X) \to \overline{M}(X)$, the proof is complete. □

Behavior in families. Here we study how $\overline{M}(X)$ behaves when X is a fiber of a flat family of varieties $\mathcal{X} \to S$. We focus on connecting the results of [Har97, BM03] to the geometry of $\overline{M}(X)$.

Quite generally, suppose we are given a morphism $\mathcal{X} \to S$ with the property that the fiber over a fixed closed point $0 \in S$ is X. If $\mathcal{X} \to S$ is flat, locally projective, and finitely presented with integral geometric fibers, then [AK80, (3.1) Theorem] states that there exists a morphism $\overline{M}(\mathcal{X}/S) \to S$ such that the geometric fiber over a point $s \in S$ is the moduli space $\overline{M}(X_s)$ of rank 1, torsion-free sheaves on the fiber X_s. We ask about the flatness properties of $\overline{M}(\mathcal{X}/S) \to S$.

The case studied in [BM03] is that case where (a) S is an irreducible k-smooth curve, (b) $0 \in S$ is a given closed point, and (c) $\mathcal{X} \subset \mathbf{P}^3_k \times_k S$ is a family of cubic surfaces such that the fiber over $0 \in S$ is X and every other fiber is smooth. (An example of such a family is $S = \mathbf{P}^1_k$, $0 =$ the origin, and $\mathcal{X}$ the family defined by the bihomogeneous polynomial $T \cdot (W^2X - Y^2Z) + S \cdot f(W, X, Y, Z)$ for f a general cubic polynomial.)

Proposition 1.8 of [Har97] states that there exists a surjection of smooth curves $S' \to S$ such that $\mathcal{X}$ contains effective generalized divisors $E_1, \ldots, E_6$, $G_1, \ldots, G_6$, and $F_{1,2}, \ldots, F_{5,6}$ that restrict to the 27 lines on a smooth fiber of $\mathcal{X} \to S$. Set $\mathcal{X}' := \mathcal{X} \times_S S'$ equal to the pullback. Then

$$\begin{aligned} \mathrm{APic}(\mathcal{X}'/S') :=& \mathrm{APic}(\mathcal{X}')/\mathrm{Pic}(S') \\ =& \text{freely generated by } \mathcal{O}(H), \mathcal{O}(E_1), \ldots, \cdots \mathcal{O}(E_6); \\ \mathrm{RAPic}(\mathcal{X}'/S') =& \text{freely generated by } \mathcal{O}(H), \mathcal{O}(E_1), \ldots, \mathcal{O}(E_5), \\ & \mathcal{O}(G_6) = \mathcal{O}(2H - E_1 - E_2 - E_3 - E_4 - E_5). \end{aligned}$$

Recall that $\mathrm{RAPic}(\mathcal{X}'/S')$ is defined to be the subgroup of rank 1 reflexive sheaves that are invertible at the generic point of X.

Hartshorne shows that sending a rank 1 reflexive sheaf $\mathcal{O}(D)$ to the reflexive hull of $\mathcal{O}(D)\mathcal{O}_X$ defines a map $\rho_0\colon \mathrm{APic}(\mathcal{X}'/S') \to \mathrm{GPic}(X)$ that maps $\mathrm{RAPic}(\mathcal{X}'/S')$ into $\mathrm{APic}(X)$. After possibly relabeling the lines, we have by [BM03, Proposition 3.10]

$$\begin{aligned} \rho_0(\mathcal{O}(E_6)) =& \mathcal{O}(D_{sing}), \\ \rho_0(\mathcal{O}(F_{i,j})) =& \mathcal{O}(D_{sing}) \text{ for } j \neq 6, \\ \rho_0(\mathcal{O}(G_6)) =& \mathcal{O}(E), \\ \rho_0(\mathcal{O}(E_i)) =& \mathcal{O}(L_i) \text{ for some ruling } L_i, \\ \rho_0(\mathcal{O}(F_{i,6})) =& \mathcal{O}(M_i) \text{ for } M_i \text{ the ruling conjugate to } L_i. \end{aligned}$$

TO BE ADDED AFTER THE ARTICLE HAS BEEN REFEREED.

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